Lecture 13 | Integer partitions – continuation

The Cohen–Remmel (meta) identity

$A = \{3,3,2,4,4,4,7\}$ – a <u>multiset</u> (finite, of natural numbers), $m_A(3)=2$, $m_A(2)=\cancel{8}\,1$, $m_A(4)=3$, $m_A(7)=1$, $m_A(100)=0$ (<u>multiplicities</u> of elements in A). <u>Norm</u> $\|A\| := \sum_{a\in A} a\, m_A(a)$.

$A \vdash \|A\|$ (A is a partition of its norm).

<u>Containment of partitions:</u> $A \supset B \iff m_A(k) \geq m_B(k)$ for every $k \in \mathbb{N}$. $\iff$ B is obtained from A by deletion of some parts. For $A \supset B$ we define $\underline{A \setminus B} = C :=$ we delete from A the parts of B.

$A_1, A_2, \ldots, A_n$ partitions, we define $\underline{A_1 \cup A_2 \cup \ldots \cup A_n} = A :=$ the multiset s.t. $m_A(k) = \max_{1\leq i\leq n} m_{A_i}(k)$, for every $k \in \mathbb{N}$.

$\underline{A_1 + A_2 + \cdots + A_n} = A :=$ — || — $m_A(k) = \sum_{i=1}^{n} m_{A_i}(k)$) — || —.

Theorem (Cohen 1981; Remmel, 1982)

Let $\mathcal{A} = (A_1, A_2, \ldots)$ and $\mathcal{B} = (B_1, B_2, \ldots)$ be infinite sequences of partitions (i.e. multisets) s.t. $\forall$ finite $I \subset \mathbb{N}$: $\left\|\bigcup_{i\in I} A_i\right\| = \left\|\bigcup_{i\in I} B_i\right\|$. $(*)$ Then for every $n\in\mathbb{N}$,

$p_{\mathfrak{A}}(n) := \#\{\lambda\vdash n \mid \forall i\in \mathbb{N}: \lambda\not\supset A_i\} = \#\{\lambda\vdash n\mid \forall i\in \mathbb{N}: \lambda\not\supset B_i\} =: p_{\mathfrak{B}}(n)$.

Proof. By the principle of inclusion-exclusion:

$$p_{\mathfrak{A}}(n) = \sum_{I\subset \mathbb{N}} (-1)^{|I|}\, \# \underbrace{\bigcap_{i\in I}\{\lambda\vdash n\mid \lambda\supset A_i\}}_{=\{\lambda\vdash n \mid \forall i\in I:\ \lambda\supset A_i\} = \{\lambda\vdash n\mid \lambda\supset \underbrace{\bigcup_{i\in I} A_i}_{=:A_I}\}}.$$

The sum (and I is finite) is effectively finite, we may assume that $i\neq j \Rightarrow A_i\neq A_j$, ~~and~~ Therefore we may restrict the sum to $I\subset N := \{i\in\mathbb{N} \mid \|A_i\|\le n\}$.

Similarly, $p_{\mathfrak{B}}(n) = \sum_{I\subset N} (-1)^{|I|}\,\#\{\lambda\vdash n\mid\lambda\supset B_I\}$ where again $B_I = \{\lambda\vdash n \mid \lambda\supset \bigcup_{i\in I} B_i\}$. ~~As before~~ Also, $A_\emptyset = B_\emptyset = \emptyset$.

As before it suffices to show that $\forall I\subset N$: $|U_I| = |V_I|$ where $U_I := \{\lambda\vdash n\mid\lambda\supset A_I\}$ and $V_I := \{\lambda\vdash n\mid \lambda\supset B_I\}$. We have the obvious bijection (by condition (*))

$$U_I\ni\lambda\longmapsto(\lambda\setminus A_I)+B_I\in V_I.$$

$\boxtimes$

Condition (*) is satisfied, if $i\neq j\Rightarrow A_i\cap A_j=\emptyset$ (i.e. $m_{A_i}(q)\,m_{A_j}(q)=0$ for $\forall q\in\mathbb{N}$) and $B_i\cap B_j=\emptyset$

and

$\|A_i\| = \|B_i\|$ for $\forall i \in \mathbb{N}$. For then $\forall$ fin. $I \subset \mathbb{N}$:
$\|\bigcup_{i\in I} A_i\| = \sum_{i\in I} \|A_i\| = \sum_{i\in I} \|B_i\| = \|\bigcup_{i\in I} B_i\|$.

Some applications:

Glaisher's identity: $\mathcal{A} = (\{d\}, \{2d\}, \{3d\}, \dots)$, $d \in \mathbb{N}$
$\mathcal{B} = (\{\underbrace{1,1,\dots 1}_{d}\}, \{\underbrace{2,2,\dots 2}_{d}\}, \{\underbrace{3,3,\dots 3}_{d}\}, \dots)$

$\forall n \in \mathbb{N}$:
$\rightsquigarrow$ n has as many partitions in parts not divisible by d as par-s with multiplicities $\le d-1$. For $d=2$ we get Euler's identity from L 12.

Squares identity: $\mathcal{A} = (\{1\}, \{4\}, \{9\}, \{16\}, \dots)$
$\mathcal{B} = (\{1\}, \{2,2\}, \{3,3,3\}, \{4,4,4,4\}, \dots)$

$\rightsquigarrow$ $\forall n \in \mathbb{N}$:
n has as many par-s s.t. no part is a square as par-s in which every part m has multiplicity $\le m-1$.

Schur's identity: $\mathcal{A} = (\{2\}, \{3\}, \{4\}, \{6\}, \{8\}, \{9\}, \{10\}, \{12\}, \{14\}, \dots)$, $\mathcal{B} = (\{1,1\}, \{3\}, \{2,2\}, \{6\}, \{4,4\}, \{9\}, \{5,5\}, \{12\}, \{7,7\}, \dots)$ $\rightsquigarrow$ $\forall n \in \mathbb{N}$: n has as many par-s in parts $\equiv \pm 1 \pmod 6$ as par-s in distinct parts $\equiv \pm 1 \pmod 3$.

$\vdots$

An identity or recurrence for $\sigma(n) = \sum_{d|n} d$.

E.g., $\sigma(2)=1+2=3$, $\sigma(3)=1+3=4$, $\sigma(4)=1+2+4=7$, $\sigma(5)=$ $=1+5=6$, $\sigma(6)=1+2+3+6=12$, ... I do not remember the identity exactly, let's derive it. The idea is to log-differentiate the pentag. identity

$$F(x) := \prod_{n=1}^{\infty}(1-x^n) = \underbrace{1+\sum_{m=1}^{\infty}(-1)^m\left(x^{\omega(m)}+x^{\omega(-m)}\right)}_{=:D}$$

where $\omega(m)=\frac{1}{2}m(3m+1)$. We apply the operator

$$-x\cdot\frac{d}{dx}\log(f(x)) = -x\,\frac{f'(x)}{f(x)}$$ on both sides:

$$\sum_{n=1}^{\infty}\frac{n x^n}{1-x^n} = \frac{\sum_{m=1}^{\infty}(-1)^{m+1}\left(\omega(m)x^{\omega(m)}+\omega(-m)x^{\omega(-m)}\right)}{D}$$

$$\left[\frac{nx^n}{1-x^n} = nx^n+nx^{2n}+nx^{3n}+\dots\right]$$

$$\Downarrow$$

$$\left(1+\sum_{m=1}^{\infty}(-1)^m\left(x^{\omega(m)}+x^{\omega(-m)}\right)\right)\sum_{n=1}^{\infty}\sigma(n)x^n$$

$$=\sum_{m=1}^{\infty}(-1)^{m+1}\left(\omega(m)x^{\omega(m)}+\omega(-m)x^{\omega(-m)}\right).$$

Comparing the coefficients of x^n on both sides we get: $n\neq\omega(\pm m) \Rightarrow \sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)$

$$-\dots = 0,$$

$n=\omega(\pm m) \Rightarrow \sigma(n)-\sigma(n-1)-\sigma(n-2)+\sigma(n-5)+\sigma(n-7)-\dots =$

$$= (-1)^{m+1}n.$$

Thus we have the

Theorem (L. Euler, 1750) $\forall n \in \mathbb{N}$:

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \ldots$$

where $\sigma(m) = 0$ for $m < 0$ and $\sigma(0) = \sigma(n-n) := n$ if n is pentagonal.

Example

$$n = 10: \quad \sigma(10) = \sigma(9) + \sigma(8) - \sigma(5) - \sigma(3) = 28 - 10 = \underline{18} \ \checkmark.$$

$$\sigma(10) = 1+2+5+10 = \underline{18}$$

$$\underbrace{\sigma(9) = 1+3+9 = 13 \qquad \sigma(8) = 1+2+4+8 = 15}_{28}$$

$$\underbrace{\sigma(5) = 1+5 = 6 \qquad \sigma(3) = 1+3 = 4}_{10}$$

$\boxed{\text{Theorem}}$ $p(n)$ (= the # of all $\lambda \vdash n$) $< \dfrac{\pi}{\sqrt{6(n-1)}} \cdot e^{c\sqrt{n}}$, $n \geq 2$

where $c = 2\sqrt{\zeta(2)} = \pi\sqrt{2/3}$ $\left(\zeta(2) = \pi^2/6\right)$.

Proof. $\displaystyle\sum_{n=0}^{\infty} p(n) t^n = F(t) := \prod_{q=1}^{\infty} \frac{1}{1-t^q}$, $t \in (0,1)$

$$F(t) > p(n)t^n + p(n+1)t^{n+1} + \ldots > p(n)(t^n + t^{n+1} + \ldots) = p(n)\frac{t^n}{1-t}$$

($p(n)$ increases). Hence

$$\log(p(n)) < \log(F(t)) - n\log t + \log(1-t)$$

$$= \sum_{q=1}^{\infty} \log\left(\frac{1}{1-t^q}\right) - n\log t + \log(1-t)$$

$$= \sum_{q=1}^{\infty} \sum_{j=1}^{\infty} \frac{t^{jq}}{j} - n\log t + \log(1-t)$$

$$= \sum_{j=1}^{\infty} t^j/(j(1-t^j)) - \text{——}\text{"}\text{——}$$

$$< \frac{t}{1-t}\sum_{j=1}^{\infty}\frac{1}{j^2} - n\log t + \log(1-t)$$

$$= \frac{t\,\zeta(2)}{1-t} - n\log t + \log(1-t)$$

$$= \frac{\zeta(2)}{u} + n\log(1+u) + \log\left(\frac{u}{1+u}\right) \text{ where } t = \frac{1}{1+u},\ u>0.$$

$$< \frac{\zeta(2)}{u} + (n-1)u + \log u, \quad (\log(1+u) < u,\ u>0)$$

$\left(\because\ \frac{1-t^j}{1-t} = 1+t+t^2+\dots+t^{j-1} > j\,t^{j-1},\ j\in\mathbb{N}\right)$. We set $u = \sqrt{\zeta(2)/(n-1)}$, apply $\exp(\cdot)$ and get the stated bound. $\boxtimes$

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the precise asymptotics of $p(n)$ is given by

Theorem (Hardy–Ramanujan, 1918) For $n\to\infty$,

$$p(n) \sim \frac{e^{\pi\sqrt{2n/3}}}{4n\sqrt{3}}.$$

THANK YOU!