L12 | Integer partitions – some identities

1

A composition $c=(a_1,\dots,a_k)\in\mathbb{N}^k$ of $n\in\mathbb{N}$ is a <u>partition of n</u> if $a_1\geq a_2\geq\dots\geq a_k$; we write $\lambda=(a_1\geq a_2\geq\dots\geq a_k)$. Another format: $1^{m_1}2^{m_2}\dots n^{m_n}$, $m_i\in\mathbb{N}_0$ (the multiplicities), $n=m_1\cdot 1+m_2\cdot 2+\dots+m_n\cdot n$. $\boxed{a_i = \text{parts}}$

Example (6), $(5\geq 1)$, $(4\geq 2)$, $(4\geq 1\geq 1)$, $(3\geq 3)$, $(3\geq 2\geq 1)$, $(3\geq 1\geq 1\geq 1)$, $(2\geq 2\geq 2)$, $(2\geq 2\geq 1\geq 1)$, $(2\geq 1\geq 1\geq 1\geq 1)$, $(1\geq 1\geq 1\geq 1\geq 1\geq 1)$ or

$6, 15, 24, 1^2 4, 3^2, 123, 1^3 3, 2^3, 1^2 2^2, 1^4 2, 1^6$

are all 11 partitions of 6. $\boxed{p(n) = \text{the \# partitions of } n;\ p(6)=11.}$

$p_o(n) :=$ # partitions of n with odd parts.

$p_d(n) :=$ ——— " ——— distinct parts.

$n=6$: $15, 3^2, 1^3 3, 1^6 \rightsquigarrow p_o(6)=4$

$6, 15, 24, 123 \rightsquigarrow p_d(n)=4$ too.

Theorem (Euler) $\forall n: p_o(n)=p_d(n)$.

Proofs. <u>Proof no. 1 by bijection</u> (a) $\forall n\in\mathbb{N}\ \exists!\, l\in\mathbb{N}, m\in\mathbb{N}_0$: l is odd & $n=l2^m$.

(b) $\forall n\in\mathbb{N}\ \exists!\, m_i\in\mathbb{N}_0: 0\leq m_0<m_1<\dots<m_k$ & $n=\sum_{i=0}^{k}2^{m_i}$.

$P_o=\{$the p.-s of n with odd parts$\}$ and

②

$P_d(n) = \{ \text{—}\ \text{"}\ \text{—}$ with distinct parts$\}$

$P_o(n) \ni \lambda = \cancel{(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k)}\ 1^{m_1} 3^{m_3} 5^{m_5} \dots,$

$\rightsquigarrow n = m_1 \cdot 1 + m_3 \cdot 3 + m_5 \cdot 5 + \dots,$ write each m_i as a sum of powers of 2 by (b) and multiply out $\rightsquigarrow n = \sum$ of different numbers, i.e. a $\varkappa \in P_d(n)$.

$P_d(n) \ni \varkappa = (\lambda_1 > \lambda_2 > \dots > \lambda_k)$, $n = \lambda_1 + \lambda_2 + \dots + \lambda_k$, write each $\lambda_i =$
$= l_i 2^{m_i}$ by (a). In the sum $n = \sum_{i=1}^{k} l_i 2^{m_i}$ ~~collect together summand with equal l_i~~ split each $l_i 2^{m_i}$ as
$l_i 2^{m_i} = \underbrace{l_i + l_i + \dots + l_i}_{2^{m_i}}$. In the resulting sum collect together ~~terms~~ the same l_i terms, $\rightsquigarrow \lambda \in P_o(n)$.

For example: $15,\ 3^2,\ 1^3 3,\ 1^6$

$$\begin{matrix} 15, & 3^2, & 1^3 3, & 1^6 \\ \updownarrow & \updownarrow & \updownarrow & \updownarrow \\ 15 & 6 & 12\ 3 & 24 \end{matrix}$$

The maps $\lambda \mapsto \varkappa$ and $\varkappa \mapsto \lambda$ are inverses of one another — we have a bijection between the sets $P_o(n)$ and $P_d(n)$.

<u>Proof no. 2 by generating functions</u> Let
$F_o(x) = \sum_{n=0}^{\infty} P_o(n) x^n$ and $F_d(x) = \sum_{n=0}^{\infty} P_d(n) x^n$. Then

③

$$F_o(x) = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} \text{ and } F_d(x) = \prod_{n=1}^{\infty} (1+x^n).$$

$$\frac{1}{1-x^{2n-1}} = 1 + x^{2n-1} + x^{2\cdot(2n-1)} + x^{3\cdot(3n-1)} + \dots$$ But

$$F_d(x) = \prod_{n=1}^{\infty}(1+x^n) = \prod_{n=1}^{\infty} \frac{1-x^{2n}}{1-x^n} = \frac{\cancel{(1-x^2)}\cancel{(1-x^4)}\cancel{(1-x^6)}\cancel{(1-x^8)}\cdots}{(1-x)\cancel{(1-x^2)}(1-x^3)\cancel{(1-x^4)}\cdots}$$

$$= \frac{1}{(1-x)(1-x^3)(1-x^5)\cdots} = \prod_{n=1}^{\infty} \frac{1}{1-x^{2n-1}} = F_o(x).$$ Thus $P_d(n) = P_o(n)$ for every $n \in \mathbb{N}$.

Proof no. 3 by PIE. (inclusion-exclusion principle)

$$P_d(n) = \sum_{\substack{I \subset [n] \\ \{i_1, i_2, \dots, i_k\}}} (-1)^{|I|} \# \overbrace{\{\underbrace{\lambda \vdash n}_{\lambda \text{ is a partition of } n} \mid \lambda \supset \{i_1, i_1, i_2, i_2, \dots, i_k, i_k\}\}}^{A_I}$$

where $A_\emptyset = \{\text{all partitions of } n\}$. Similarly,

$$P_o(n) = \sum_{\substack{I \subset [n] \\ \{i_1, \dots, i_k\}}} (-1)^{|I|} \# \overbrace{\{\lambda \vdash n \mid \lambda \supset \{2i_1, 2i_2, \dots, 2i_k\}\}}^{B_I}.$$

and $B_\emptyset = A_\emptyset$ $\forall I \subset [n]$ there is a bijection $A_I \longleftrightarrow B_I$, simply $i_j, i_j \longleftrightarrow 2i_j$, i.e. $\forall j = 1, 2, \dots, k$ take two parts i_j and replace them with single part $2i_j$. Thus $\forall I: |A_I| = |B_I|$ and also $P_d(n) = P_o(n)$, for every $n \in \mathbb{N}$. ⊠

$\boxed{\text{theorem (Euler's pentagonal identity)}}$ The following elementarily equivalent results hold true. 4

① $\prod_{n=1}^{\infty}(1-x^n) = (1-x)(1-x^2)(1-x^3)(1-x^4)\cdots$

$= \sum_{m=-\infty}^{m=+\infty}(-1)^m x^{\overbrace{\frac{m}{2}(3m+1)}^{\omega(m)}} = 1+\sum_{m=1}^{\infty}(-1)^m(x^{\omega(m)}+x^{\omega(-m)})$.

② $\#\{\lambda \vdash n \mid \lambda\in P_d(n)\ \&\ \lambda$ has an even # of parts$\}$ −
− $\#\{$ —— | —— odd —— | —— $\}$ =

$= \begin{cases} 0 & \dots\ n\neq\omega(\pm m) \\ (-1)^m & \dots\ n=\omega(\pm m)\end{cases}$ n is a pentagonal number. ($m\in\mathbb{Z}$)

③ $m\in\mathbb{N}$, $\omega(\pm m) = 1,2,5,7,12,15,22,26,\dots$ Then $\forall n\in\mathbb{N}$: $p(n)=p(n-1)+p(n-2)-p(n-5)-p(n-7)+p(n-12)+\dots$ where $p(x)=0$ for $x<0$ and $p(0)=1$.

$\boxed{\text{Proof.}}$ $[x^n]\prod_{q=1}^{\infty}(1-x^q)$ = the difference in ②, so ① $\Leftrightarrow$ ②. Since $\sum_{n=0}^{\infty}p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}$,

(coeff. of x^n)

$1=\prod_{q=1}^{\infty}(1-x^q)\sum_{n=0}^{\infty}p(n)x^n$, which gives that ① $\Leftrightarrow$ ③.

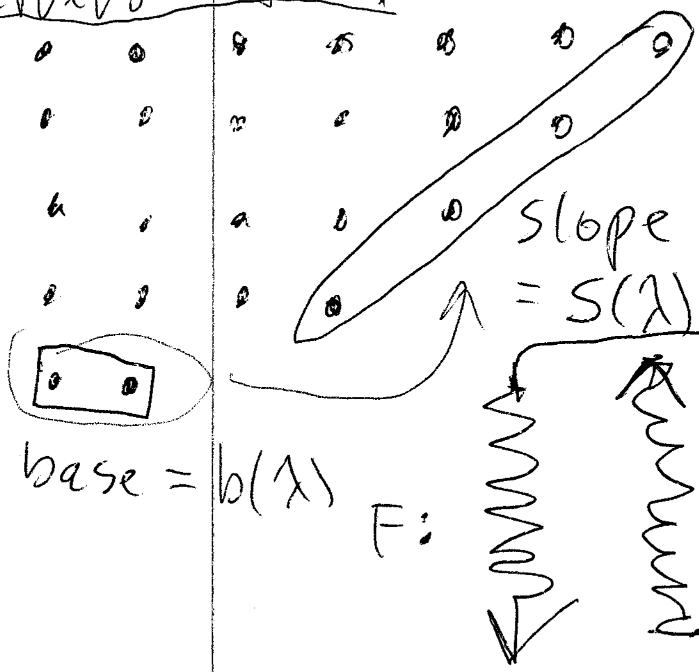
We prove bijectively ②. We define a (partial) map

$F: P_d(n) \to P_d(n)$ a partial map

s.t. (i) F is an involution ($F \circ F = id$, $F = F^{-1}$), (ii) the # of parts in $F(\lambda)$ = the # of parts in $\lambda \pm 1$, and (iii) for $n \neq \omega(\pm m)$ the map F is always defined, but for $n = \omega(\pm m)$ it is undefined on exactly one $\bar{\lambda} \in P_d(n)$ (*)

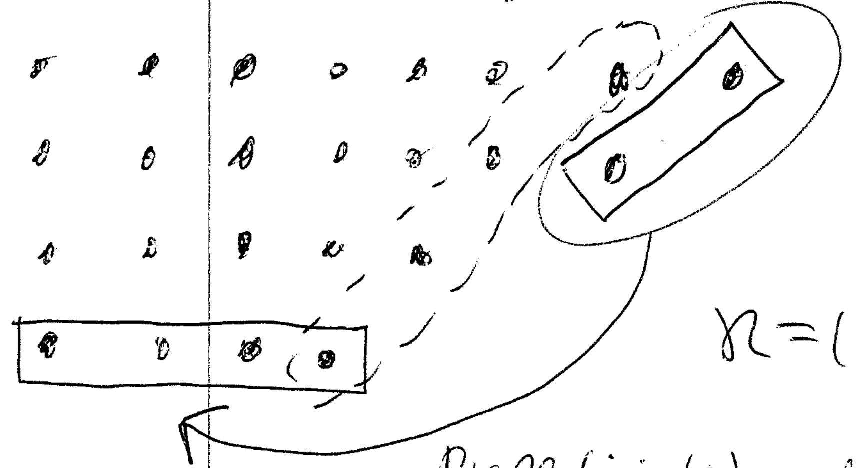
$\boxed{n = 24}$ Definition by example: $\lambda = (7 > 6 > 5 > 4 > 2)$

Ferrers diagram:



If $|b(\lambda)| \leq |s(\lambda)| \Rightarrow$ do this. If $|b(\lambda)| > |s(\lambda)| \Rightarrow$ do this.

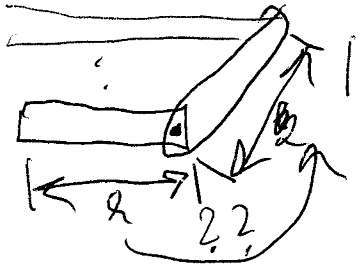
(*) and the # of parts of $\bar{\lambda}$ has parity $(-1)^m$.



$\mu = (8 > 7 > 5 > 4)$

Properties (i) and (ii) are clear. As

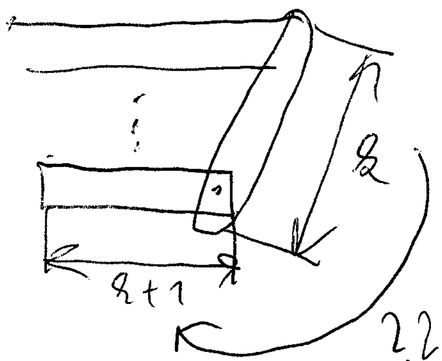
for (iii), when the above operation F does not work? Iff $\bar{\lambda}$, =



$= \quad |b(\lambda)| = |s(\lambda)| = q$ then $n = q + (q+1) + (q+2) + \dots + (2q-1) = \frac{2q(2q-1)}{2} - \frac{q(q-1)}{2}$

$$= \frac{4q^2 - 2q - q^2 + q}{2} = \frac{3q^2 - q}{2} = \frac{q}{2}(3q-1) = \omega(-q).$$

Or if $|b(\lambda)| = q+1$, $|s(\lambda)| = q$ then



$= \quad n = (q+1) + (q+2) + \dots + 2q =$

$$= \frac{2q(2q+1)}{2} - \frac{q(q+1)}{2} =$$

$$= \frac{4q^2 + 2q - q^2 - q}{2} = \frac{3q^2 + q}{2} = \frac{q}{2}(3q+1) = \omega(q).$$

$\boxtimes$

THANK YOU!