L7 (April 17, 2020) $\boxed{\text{Theorem}}$ ($\Longleftrightarrow$ for LRS)

A sequence $(a_n)=(a_1,a_2,\dots)\subset\mathbb{C}$ satisfies a recurrence (*) $a_{n+k}=\sum_{i=0}^{k-1} c_i a_{n+i}$ $\forall n\in\mathbb{N}$ (where $k\in\mathbb{N}_0$ — for $k=0$ the RHS $:=0$ — and $c_0,\dots,c_{k-1}\in\mathbb{C}$)

$\Updownarrow$

The gen. function $A(x)=\sum_{n=1}^{\infty} a_n x^n$ is rational, $A(x)=\frac{p(x)}{q(x)}$ for some polynomials $p,q\in\mathbb{C}[x]$ with $q(0)\neq 0$ and $\boxed{\deg(p)\le\deg(q)}$ $\boxed{\text{or } p\equiv 0}$.

Proof. $\Rightarrow$. We assume that (a_n) satisfies recurrence (*). Then

$$0=\sum_{n=1}^{\infty} x^{n+k}\Big(a_{n+k}-\sum_{i=0}^{k-1}c_i a_{n+i}\Big)=A(x)-B_k(x)-\sum_{i=0}^{k-1}c_i x^{k-i}\big(A(x)-B_i(x)\big),$$

where $B_0(x)=0$, $B_1(x)=a_1x$, $B_2(x)=a_1x+a_2x^2$, ..., $B_k(x)=a_1x+a_2x^2+\dots+a_kx^k$. We solve equation [...] for $A(x)$ and get that

$$A(x)=\frac{\overbrace{B_k(x)+c_{k-1}xB_{k-1}(x)+\dots+c_0x^kB_0(x)}^{\deg\le k \text{ or } \equiv 0}}{\underbrace{1-c_{k-1}x-\dots-c_0x^k}_{\deg=k}}.$$

We

Call the denominator $1-c_{k-1}x-c_{k-2}x^2-\dots-c_0x^k$ the <u>characteristic polynomial</u> of the recurrence.

the polynomial $x^k-c_{k-1}x^{k-1}-c_{k-2}x^{k-2}-\dots-c_1x-c_0$ that is reciprocal to ~~the~~

$\Leftarrow$. We assume that $A(x)=\sum_{n=1}^{\infty}a_nx^n=\frac{p(x)}{q(x)}$ for some $p,q\in\mathbb{C}[x]$ with $q(0)\neq 0$ (so $q(x)$ is a unit in $\mathbb{C}[[x]]$) and $\deg(p)\leq\deg(q)$ or $p\equiv 0$. If $p\equiv 0$ then $A(x)=0$, $(a_n)=(0,0,0,\dots)$ and (a_n) satisfies e.g. the recurrence $a_{n+1}=a_n$. So let $p\not\equiv 0$ and $\deg(p)\leq\deg(q)=:k\in\mathbb{N}_0$. Then $q(x)A(x)=p(x)$. We may assume that $q(x)=1-c_1x-c_2x^2-\dots-c_kx^k$, $c_k\neq 0$ [Exercise: why may we assume this?] and therefore

$$(1-c_1x-c_2x^2-\dots-c_kx^k)\sum_{n=1}^{\infty}a_nx^n=\overbrace{p(x)}^{\deg\leq k}.$$

For $n=1,2,\dots$ we compare the coefficient of x^{n+k} on the left, which is

$a_{n+k}-c_1a_{n+k-1}-\dots-c_ka_n$, with the same coeff.

on the right which is 0. Thus $\forall n \in \mathbb{N}$,

$$a_{n+k} = \sum_{i=0}^{k-1} c_{k-i}\, a_{n+i}, \quad c_k \neq 0.$$

This is the recurrence (*), with $c_k \neq 0$, only the coeff-s are now indexed in reverse. $\boxtimes$

So, to restate the theorem (I made some corrections and it may not be clearly legible):

$(a_n) = (a_1, a_2, \dots) \subset \mathbb{C}$ satisfies for $\forall n \in \mathbb{N}$ a recurrence $\sum_{i=0}^{k-1} c_i a_{n+i} = a_{n+k}$ (for constants $k \in \mathbb{N}_0$ and $c_i \in \mathbb{C}$ with $c_0 \neq 0$) $\Longleftrightarrow$ $A(x) = \sum_{n=1}^{\infty} a_n x^n = \frac{p(x)}{q(x)}$ for some $p, q \in \mathbb{C}[x]$ with $q(0) \neq 0$ and $\deg(p) \leq \deg(q)$ or p identically zero.

The condition on degrees is needed for the implication $\Leftarrow$ to hold. For example, $\frac{x^2}{1-x} = x^2 + x^3 + \dots$ gives sequence $(a_1, a_2, \dots) = (0, 1, 1, 1, \dots)$ which is not a LRS, does not satisfy any recurrence of the form (*). [Exercise: why? this was already an exercise before.]

The last debt to pay (from the 4 proofs that (C_n) is not a LRS) is to prove the equivalence: $(a_n) \subset \mathbb{C}$ is a LRS $\iff$ a_n has an expression in the form of a ~~power~~ power sum. We introduce power sums by means of[(1)] the well known **Binet's formula** for the Fibonacci numbers. The Fibonacci numbers $(F_n) \subset \mathbb{N}$, $(F_n) = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \dots)$ are given by the recurrence $F_{n+2} = F_{n+1} + F_n$. The journal Fibonacci Quaterly is devoted to them. By the previous ~~proof~~ equivalence,

$$F(x) = \sum_{n=1}^{\infty} F_n x^n = \frac{p(x)}{1-x-x^2}$$

for some $p \in \mathbb{C}[x]$ with $\deg(p) \le 2$. Since $(1-x-x^2)F(x) = (1-x-x^2)(x+x^2+2x^3+\dots) = x + 0x^2 + 0x^3 + \dots$, $p(x) = x$ and

$$\boxed{F(x) = \frac{x}{1-x-x^2}.}$$

We factorize the denominator as $(*)$ $1-x-x^2 = (1-\alpha x)(1-\beta x)$ and ex-

(1) and the method of proof of the $\iff$.

press the right side in partial fractions: (5)

$$(+)\quad \frac{x}{1-x-x^2} = \frac{\gamma}{1-\alpha x} + \frac{\delta}{1-\beta x}$$ where $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ (they are in fact real). Let's determine these constants. From (•) we have that $\alpha+\beta = 1$ and $\alpha\beta = -1$. So $\alpha(1-\alpha) = -1$, $\alpha^2-\alpha-1=0$, $\alpha_{1,2} = \frac{1}{2}(1 \pm \sqrt{1+4\cdot 1})$.

Hence $\underline{\alpha = \frac{1+\sqrt{5}}{2}}$ and $\underline{\beta = \frac{1-\sqrt{5}}{2}}$ or they ~~to~~ may be switched. From (+) we get that

$\gamma(1-\beta x) + \delta(1-\alpha x) = x$, hence $\underline{\gamma+\delta=0}$ and $\underline{\gamma\beta+\delta\alpha = -1}$. Thus $\delta = -\gamma$ and $\gamma(\beta-\alpha) = -1$ and $\boxed{\gamma = \frac{1}{\alpha-\beta} = \frac{1}{\sqrt{5}}}$ and

$\boxed{\delta = -\frac{1}{\sqrt{5}}}$. Since for $u, v \in \mathbb{C}$ we have in the ring $\mathbb{C}[[x]]$ of formal power series the identity $\frac{u}{1-vx} = u\sum_{n=0}^{\infty} v^n x^n$, we get from (+)

the well known Binet's formula:

$$F(x) = \sum_{n\geq 1} F_n x^n = \frac{x}{1-x-x^2} = \frac{\gamma}{1-\alpha x} + \frac{\delta}{1-\beta x} = \sum_{n=0}^{\infty} \gamma \alpha^n x^n + \sum_{n=0}^{\infty} \delta \beta^n x^n,$$

therefore $\forall n \in \mathbb{N}$,

$$\boxed{F_n = \gamma \alpha^n + \delta \beta^n = \underbrace{\frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)}_{\text{this is a particular case of power sum}}}$$

this is a particular case of power sum. The general equivalence $(a_n) \subset \mathbb{C}$ is a LRS $\iff$ a_n = a power sum will be proven in the next lecture.

PS **Exercise** Show that $\forall n \in \mathbb{N}$,

$$F_n = \left\| \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n \right\|.$$

Here for $\alpha \in \mathbb{R}$, we denote by $\|\alpha\| \in \mathbb{Z}$ the integer closest to α.