L6 (April 3, 2020) | Let me still return to the proof of the equivalence ①

$\underbrace{(a_n) \subset \mathbb{C} \text{ holon.} \iff \sum_{n=0}^{\infty} a_n x^n \text{ holon.}}$ in the previous lecture. In the proof of $\Leftarrow$ it is simpler to take any $i_0 \in \{0, 1, \dots, k\}$ s.t. $J_{i_0} \neq \emptyset$ (i.e. $q_{i_0}(x) \neq 0$) and then any $j_0 \in J_{i_0}$. It is more precise to set $m = i_0 - j_0$ and to look at the coeff. of $a_{n+m}$ where $n$ is a formal variable and $m \in \mathbb{Z}$ is fixed. This coeff. is a sum of several (and at least one) $\underbrace{\phantom{x}}_{\text{(nonzero)}}$ elements of $\mathbb{C}[n]$.

Crucially, in this sum the summands, $\neq 0$ polynomials from $\mathbb{C}[n]$, have mutually distinct degrees and therefore the sum is a $\neq 0$ element of $\mathbb{C}[n]$ (the summand with the max. degree cannot be cancelled). $\boxtimes$

I will pay two debts from L4, the first one being ~~that~~ another equivalence: a sequence $(a_n) \subset \mathbb{C}$ is a linear recurrence sequence (LRS) $\iff$ the OGF $A(x) = \sum_{n=0}^{\infty} a_n x^n$ is rational, $= \frac{p(x)}{q(x)}$ precisely with $p, q \in \mathbb{C}[x]$, $q(0) \neq 0$.

First we define LRS. A sequence $(a_n) =$

$=(a_0, a_1, a_2, \dots) \subset \mathbb{C}$ is a <u>LRS</u> if $\exists k \in \mathbb{N}_0$ (2)

$\exists c_1, c_2, \dots, c_k \in \mathbb{C}$, $c_k \neq 0$, s.t. for every $n \in \mathbb{N}_0$,

$$\boxed{a_{n+k} = c_1 a_{n+k-1} + c_2 a_{n+k-2} + \dots + c_k a_n}.$$

For $k=0$ the RHS is $\emptyset$, we define it as $0$ and get the <u>zero sequence</u> with $a_n = 0$ for every $n \in \mathbb{N}_0$. This is the same definition as we gave earlier(*), only now the recurrence holds for every $n \in \mathbb{N}_0$ and not just for every $n > n_0$. A well known example of a LRS is the <u>Fibonacci numbers $(F_n)$</u> with $F_0 = F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, so $(F_n) = (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots)$. $(a_n) = (1, 0, 1, 0, 1, 0, \dots)$ is also a LRS, given by the recurrence

<u>$a_{n+2} = a_n \; (= 0 \cdot a_{n+1} + 1 \cdot a_n)$.</u> Exercise for you

Prove that the eventually constant sequence $(a_n) = (1, 0, 2, 3, 4, 2020, 2020, 2020, 2020, \dots)$, $a_n = 2020$ for every $n \geq 5$, is <u>not</u> a LRS.

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The precise version of the above equivalence is as follows.

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(*) For the 4 proofs that the Catalan #s $(C_n)$ is not a LRS.

$\boxed{\text{Theorem}}$ $(a_n) = (a_0, a_1, \dots) \subset \mathbb{C}$ is a LRS

$$\Updownarrow$$

$\exists$ polynomials $p, q \in \mathbb{C}[x]$ s.t. $q(0) \neq 0$, $p(x) = 0$ or $\deg(p) < \deg(q)$, and $\sum_{n=0}^{\infty} a_n x^n = \frac{p(x)}{q(x)}$.

Before plunging in the proof we have to explain, of course, what exactly does the equality mean. It is an equality in the ring $\mathbb{C}[[x]]$ of formal power series. In the equality we have three elements of it, $A(x) = \sum_{n=0}^{\infty} a_n x^n$, $p(x)$ and $q(x)$, where $q(x) \neq 0$, and the equality claims that $A(x) = p(x) \cdot q(x)^{-1}$. Here $q(x)^{-1}$ is the multiplicative inverse in $\mathbb{C}[[x]]$ to $q(x)$. So how are these inverses defined? The elements of $\mathbb{C}[[x]]$ are the formal linear combinations ((*)) $\sum_{n=0}^{\infty} b_n x^n$ of the powers $x^n$, $n \in \mathbb{N}_0$, with complex coefficients $b_n \in \mathbb{C}$. Addition and multiplication are defined in $\mathbb{C}[[x]]$ as follows.

(*) $x$ is a formal variable.

((an) is now a generic sequence, not the one ~~from~~ in the above theorem) $\sum_{n\geq 0} a_n x^n + \sum_{n\geq 0} b_n x^n :=$

$:= \sum_{n\geq 0} (a_n + b_n) x^n$ and

addition in $\mathbb{C}$

addition in $\mathbb{C}[[x]]$, defined by this equality

$$\sum_{n\geq 0} a_n x^n \cdot \sum_{n\geq 0} b_n x^n := \sum_{n=0}^{\infty} \underbrace{\left(\sum_{k=0}^{n} a_k b_{n-k}\right)}_{\text{multiplication and addition in } \mathbb{C}.} x^n.$$

multiplication in $\mathbb{C}[[x]]$

This is also sometimes called the Cauchy product of (formal) power series.

**Exercise** Show that $(\mathbb{C}[[x]], +, \cdot)$, with the neutral elements $0 = 0\cdot x^0 + 0\cdot x^1 + \ldots$ and $1 = 1\cdot x^0 + 0\cdot x^1 + 0\cdot x^2 + \ldots$) is a ring (i.e. satisfies all axioms of a ring)

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**Proposition** $\mathbb{C}[[x]]$ is an integral domain (*)

Proof: Let $\sum_{n\geq 0} a_n x^n$, $\sum_{n\geq 0} b_n x^n$ be $\neq 0$ elements of $\mathbb{C}[[x]]$. Let $m_0$ be the minimum $n$ s.t. $a_n \neq 0$ and $n_0$ be —"— $b_n \neq 0$. Then $\sum_{n\geq 0} a_n x^n \cdot \sum_{n\geq 0} b_n x^n =$

(*) R is an i.d.: $a, b \in R$, $ab = 0_R \Rightarrow a = 0_R \vee b = 0_R$

$= \underbrace{a_{n_0} b_{n_0}}_{\neq 0} X^{n_0+n_0} + \cdot X^{n_0+n_0+1} + \dots \neq 0$ in $\mathbb{C}[[X]]$.

Thus $\mathbb{C}[[X]]$ is an integral domain ⊠

The minimum $n \in \mathbb{N}_0$ s.t. $a_n \neq 0$ in a $0 \neq A(x) = \sum_{n \geq 0} a_n x^n \in \mathbb{C}[[x]]$ is in fact called the order of $A(x)$ and denoted $\mathrm{ord}(A(x))$. We set $\mathrm{ord}(0) = -\infty$.

**Exercise** Prove that for every $A, B \in \mathbb{C}[[x]]$, $\mathrm{ord}(A \cdot B) = \mathrm{ord}(A) + \mathrm{ord}(B)$.

Now we come to the question of units in the ring $\mathbb{C}[[x]]$ (i.e. the invertible elements) which has to be clarified before we can prove the above theorem.

**Proposition** $A(x) \in \mathbb{C}[[x]]$ is a unit if and only if $\mathrm{ord}(A(x)) = 0$, i.e. $A(x) = a_0 + a_1 x + a_2 x^2 + \dots$ with $\boxed{a_0 \neq 0.}$

**Proof.** Suppose that $A(x)$ is a unit, there is a $B(x) \in \mathbb{C}[[x]]$ s.t. $A(x)B(x) = 1$ $(= 1x^0 + 0x^1 + \dots)$. By the exercise ↑, $\mathrm{ord}(A) + \mathrm{ord}(B) = \mathrm{ord}(1) = 0$, which implies that $\mathrm{ord}(A) = \mathrm{ord}(B) = 0$ (because the values of $\mathrm{ord}(\cdot)$ are $\mathbb{N}_0 \cup \{-\infty\}$). Suppose that $\mathrm{ord}(A) = 0$. We are looking for a $B(x) = b_0 + b_1 x + \dots \in \mathbb{C}[[x]]$ s.t. $A \cdot B = 1$.

B'' This is equivalent with the infinite linear non-homogeneous system of equations (*)

$a_0 b_0 = 1$, $a_0 b_1 + a_1 b_0 = 0$, $a_0 b_2 + a_1 b_1 + a_2 b_0 = 0 \dots$

where $a_0, a_1, a_2, \dots \in \mathbb{C}$ are given and $b_0, b_1, b_2, \dots$ are the unknowns. Since $a_0 \neq 0$, we see that the system has a unique solution $b_0, b_1, \dots \in \mathbb{C}$:

$b_0 = \frac{1}{a_0}$, $b_1 = \frac{1}{a_0}(-a_1 b_0)$, $b_2 = \frac{1}{a_0}(-a_1 b_1 - a_2 b_0)$, and so on. Thus $A$ has an inverse $B$ and is a unit. ☒

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Now the condition $q(0) \neq 0$ in the theorem above is clear. Of course, we take $\mathbb{C}[x]$ as contained in $\mathbb{C}[[x]]$. Exercise In $\mathbb{C}[[x]]$, $\frac{1}{1-x} = ?$

And $\frac{1}{1+x} = ?$

Now we are theoretically prepared ~~for the proof of~~ to prove the above ($\Longleftrightarrow$) theorem on LRS but of course do not have time for it. ~~and the~~ I will present the proof in the next lecture. See you next week.

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(*) In better ~~[illegible]~~ English ~~[illegible]~~: inf. system of nonh. lin. eq-s,