(L4) (March 20, 2020 (!)). (1)

I defer resolving the remark from the previous lecture (non-triv. $\Rightarrow$ non.-triv.) to the next lecture. Instead in this lecture I give several proofs of the next theorem. $\boxed{\text{Theorem}}$ The Catalan numbers do not satisfy any linear recurrence with constant (0) coefficients, the relation $\boxed{C_n = a_1 C_{n-1} + a_2 C_{n-2} + \dots + a_k C_{n-k}}$ for $\forall n > n_0$ and some constants $c_i \in \mathbb{C}$ is impossible.

(Proof 1.) By 2-adic valuation.

For a nonzero $\alpha = \frac{a}{b} \in \mathbb{Q}$ and a prime number p, the p-adic order (valuation) $\nu_p(\alpha) \in \mathbb{Z}$ is defined by $\nu_p(\alpha) = \nu_p(a) - \nu_p(b)$ and for $n \in \mathbb{Z} \setminus \{0\}$, $\nu_p(n) = \max k \in \mathbb{N}_0$ s.t. p^k divides n. We set $\nu_p(0) = +\infty$. By the FTArit., $\nu_p(\alpha\beta) = \nu_p(\alpha) + \nu_p(\beta)$. Also, $\nu_p(\alpha+\beta) \geq \min(\nu_p(\alpha), \nu_p(\beta))$ where for $\nu_p(\alpha) \neq \nu_p(\beta)$ the inequality holds as equality. We assume for contradiction that the relation $\boxed{\dots}$ holds for every $n > n_0$ and some co-eff-s $a_1, \dots, a_k \in \mathbb{Q}$ [in the thm. $a_i \in \mathbb{C}$, it turns out that $a_i \in \mathbb{C} \Leftrightarrow a_i \in \mathbb{Q}$, I return to this in the next lecture]. Thus for every $n > n_0$,

$$(*) \quad a_0 c_n + a_1 c_{n-1} + \dots + a_k c_{n-k} = 0$$

where $a_i \in \mathbb{Q}$ and $a_0 = -1$. Let $j \in \{0, 1, \dots, k\}$ have minimum value $\nu_2(a_j)$, so $\nu_2(a_j) \le \nu_2(a_0) = 0$. We take $N \in \mathbb{N}$ so large that $N > n_0, 2k$ and s.t. $N-j$ is a power of 2. Then no member $N-i$, $i \neq j$, $i \in \{0, 1, \dots, k\}$ is a power of 2. As we know, c_{N-j} is odd but all other c_{N-i}, $i \neq j$, in $(*)$ are even. Thus the minimum (with $n = N$)

$$\min_{0 \le i \le k} \nu_2(a_i c_{N-i}) = \min_{0 \le i \le k} (\nu_2(a_i) + \nu_2(c_{N-i}))$$

is attained for the unique index $i = j$. We have the contradiction

$$+\infty = \nu_2(0) = \nu_2\Big(\sum_{i=0}^{k} a_i c_{N-i}\Big) = \nu_2(a_j c_{N-j}) =$$

$$↯ \quad = \nu_2(a_j) \le 0 \quad \boxtimes$$

Proof 2. <u>By irrationality of the OGF $C(x)$</u>

It is well known and true that every lin. rec. seq. (with constant coefficients) has OGF that is rational, a ratio of two polynomials. ~~(*) would imply that~~ [I return to this result in the next lecture as well.] Thus (*) would imply that, since $C(x) = \frac{1}{2}(1-\sqrt{1-4x})$, ~~(with coeffs in~~ we have

$$C(x) = \sum_{n\ge 1} c_n x^n$$

$$\sqrt{1-4x} = \frac{a(x)}{b(x)}$$ for some polynomials $a, b \in \mathbb{C}[x]$, both non-zero. Hence $(1-4x)\,b(x)^2 = a(x)^2$. But this is an impossible equality, $\deg(\mathrm{LHS})$ is odd and $\deg(\mathrm{RHS})$ is even. $\boxtimes$

Proof 3. <u>By asymptotic analysis.</u> Asymptotically, $c_n \sim c\, n^{-3/2} 4^n$ (by Stirling formula). But this asymptotics turns out incompatible with asymptotics of terms of a lin. rec. seq. with const. coeff-s. Namely, ~~the latter sequence~~

If (c_n) satisfies (0) then for $\forall n > n_0$)

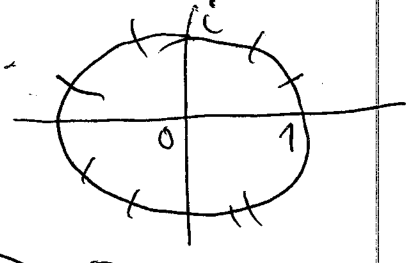
(**) $c_n = \sum_{j=1} p_j(n)\gamma_j^n$ [in the next lecture]

where $r \in \mathbb{N}_0$, $\gamma_j \in \mathbb{C}$ are distinct $\neq 0$ numbers and $p_j(x)$ are $\neq 0$ polynomials. (**) If $\max_{1\le j\le r} |\gamma_j|$ is attained for unique j, then (**) is clearly incompatible with $c_n \sim c n^{-3/2} 4^n$, but the problem is when the maximum is not attained for only one j.

Recall the <u>Vandermonde determinant</u>: if $t_1, t_2, \dots, t_n$ are variables then $\det(t_i^{j-1})_{i,j=1}^{n} =$

$$= \prod_{1\le i<j\le n} (t_j - t_i).$$ With the help of it we prove the next lemma.

$\boxed{\text{Lemma}}$ Let $t \in \mathbb{N}$ and for $j = 1, 2, \dots, t$, $d_j \in \mathbb{C}\setminus\{0\}$, $z_j \in \mathbb{C}$ with $|z_j| = 1$, and let the t numbers z_j be distinct.

z_j: 

Then $\limsup_{n\to\infty} \left| \sum_{j=1}^{t} d_j z_j^n \right| > 0.$

<u>Proof.</u> For contrary let the lim sup be 0, which

means that $\lim_{n\to\infty} \underbrace{\sum_{j=1}^{t} d_j z_j^n}_{=:v(n)} = 0$. Thus for every $r \in \mathbb{N}$ there is an $n_r \in \mathbb{N}$ s.t. for $m = 1, 2, \ldots, t$,

$$|v(n_r + m)| < \frac{1}{r}.$$

From the linear system

$$v(n_r+1) = \sum_{j=1}^{t} d_j z_j^{n_r+1}$$

$$\vdots$$

$$v(n_r+t) = \sum_{j=1}^{t} d_j z_j^{n_r+t}$$

we express $d_1, \ldots, d_t$ in terms of by the Cramer rule:

$$d_j = \frac{\det(M(j))}{\det\left(z_l^{n_r+m}\right)_{m,l=1}^{t}}$$

where the $t \times t$ matrix $M(j)$ arises from the matrix by replacing the j-th column with the column $\begin{pmatrix} v(n_r+1) \\ v(n_r+2) \\ \vdots \\ v(n_r+t) \end{pmatrix}$. By the def. of determinant, the Δ ineq., and the assumption on ~~…~~ $v(n_r+m)$ we have the bound $|\det(M(j))| \le \frac{t!}{r}$. If we take out $z_l^{n_r+1}$ from the l-th column of the matrix, it becomes a Vandermonde matrix:

In the l-th column there remain the powers z_l^{m-1}, $m=1,2,\dots,t$. Since the z_l are distinct,

$$\left|\det\left(z_l^{n_k+m}\right)_{m,l=1}^{t}\right| = \prod_{1\le l<l'\le t} |z_{l'}-z_l| =: d > 0,$$

– a $\neq 0$ constant independent of k. Thus, for every $j=1,2,\dots,t$,

$$|d_j| \le \frac{t!}{dk} \to 0,\ k\to\infty.$$

Hence all $d_j=0$, which is a $\lightning$. (of Lemma) $\boxtimes$

From $(**)$ we have that

$$c_n = (d_1 z_1^n + \dots + d_t z_t^n)\, n^k \gamma^n + O(n^{k-1}\gamma^n)$$

where $d_j \in \mathbb{C}\setminus\{0\}$, $t\in\mathbb{N}$, $k\in\mathbb{N}_0$, z_j are distinct and lie on the complex unit circle $\mathbb{T}$, and $\gamma>0$ is the maximum modulus of γ_i in $(**)$. By the above Lemma, $\limsup_{n\to\infty}\left|\frac{c_n}{n^k\gamma^n}\right|>0$, which indeed contradicts the asymptotics $c_n \sim c n^{-3/2} 4^n$. $\boxtimes\boxtimes$

The fourth proof next time (+ the debts).