Combinatorial counting, lecture 2

$C = C(x) = \sum_{n \geq 1} c_n x^n$, where $c_n = |T_n| = $ # of (mutually distinct) up trees with n vertices, is the OGF (ordinary gener. function) of the sequence $(c_n) = (c_1, c_2, \dots)$. Last time we derived the quadratic equation

$$C^2 - C + x = 0.$$

Thus $C = \frac{1}{2}(1 - \sqrt{1-4x})$ and $c_n = \frac{(-1)^{n+1}}{2}\binom{1/2}{n}4^n$ 1)

So $c_n = (-1)^{n+1} \dfrac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})\cdots(\frac{1}{2}-n+1)}{n! = 1\cdot 2 \cdot \dots \cdot n} 4^n =$

$$= \frac{1\cdot 3\cdot 5\cdot \dots \cdot (2n-3)}{2^{n+1}\cdot n!} 4^n = \frac{(2n-2)!}{n!\,(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}.$$ 2)

Thus we have even 2 expressions, 1) and 2), of c_n in terms of the binom. coefficients. From 1) we immediately get quite precise estimate on c_n: if $\alpha \in \mathbb{C}$, $|\alpha| \leq 1$, $n \in \mathbb{N}$, then

$$\left|\binom{\alpha}{n}\right| = \prod_{i=0}^{n-1}\left|\frac{\alpha - i}{i+1}\right| \leq 1\cdot 1 \cdot \dots \cdot 1 = 1.$$

For $\alpha = \frac{1}{2}$ we have even more precisely that $\forall n \in \mathbb{N}$ ($\emptyset$ product $:= 1$)

$$\frac{1}{4n^2} \leq \frac{1}{4}\cdot\frac{1\cdot 2 \cdots (n-2)}{n!} \leq \left|\binom{\frac{1}{2}}{n}\right| \leq \frac{1}{2}\cdot\frac{1\cdot 2\cdots (n-1)}{n!} = \frac{1}{2n}$$

Thus 1) gives that $\forall n \in \mathbb{N}$: $\boxed{\frac{4^n}{8n^2} < C_n \le \frac{4^n}{4n}}$. ②

(Exercise): Deduce similar bounds (with multipl. gap roughly n between the lower and the upper bound) from the formula (2). Hint: the binomial theorem.

A better (than the basic one) recurrence for C_n.

$n \ge 2$:

$$\frac{C_n}{C_{n-1}} = \frac{\frac{1}{n}\binom{2n-2}{n-1}}{\frac{1}{n-1}\binom{2n-4}{n-2}} = \frac{n-1}{n} \cdot \frac{(2n-2)(2n-3)}{(n-1)(n-1)} = \frac{4n-6}{n}.$$

Thus $\boxed{C_1 = 1 \text{ and } C_n = \frac{4n-6}{n} \cdot C_{n-1} \text{ for } n \ge 2}$ Now I show how to deduce this recurrence from the quadratic equation for $C = C(x)$, without solving it. (exercise: solve this)

$$C^2 - C + x = 0$$

$$2CC' - C' + 1 = 0, \text{ so } \boxed{C' = \frac{1}{1-2C}} =$$

$$= \frac{-\frac{C}{2} + \frac{1}{4}}{(1-2C)(-\frac{C}{2} + \frac{1}{4})} = \frac{-\|-}{C^2 - C + \frac{1}{4}} = \frac{-\|-}{\frac{1}{4} - x} \text{ and}$$

$$\boxed{(1-4x)C' + 2C - 1 = 0.} \quad (*)$$

Comparing coef-s of x^n we ~~derive~~ obtain a recurrence for C_n: $n \in \mathbb{N}$, $[x^{n-1}]\,LHS = [x^{n-1}]\,RHS$ gives that

(3) $nc_{n-1} - 4(n-1)c_{n-1} + 2c_{n-1} = 0$, thus $c_n = \frac{4n-6}{n} c_{n-1}$.

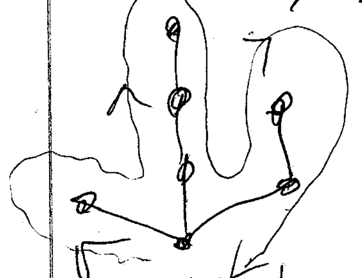
A combinatorial proof of the formula $c_n = \frac{1}{n}\binom{2n-2}{n-1}$

$D=(d_1,d_2,\dots,d_{2n}) \in \{-1,1\}^{2n}$ is a <u>Dyck word</u> of <u>size $n\in\mathbb{N}_0$</u> if $d_1+d_2+\dots+d_j \geq 0$ $\forall\, 1\leq j\leq 2n$ and $d_1+d_2+\dots+d_{2n}=0$ (i.e., D has n 1's and n -1's)

$\mathcal{D} := \{\text{all Dyck words}\}$, $\mathcal{T}=\{\text{all up trees}\}$

Proposition $\exists$ bijection $f:\mathcal{T}\to\mathcal{D}$, $|f(T)|+1=|T|$ (f decreases size by 1).

<u>Proof.</u>



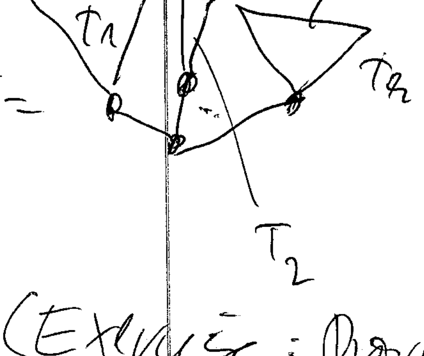
$\uparrow = 1$

$\downarrow = -1$

$T \longmapsto D = f(T)$, $|D| = 6$.

$|T| = 7$ $\quad = (1,-1,1,1,1,-1,-1,-1,1,1,-1,-1)$

Or recursively: $\bullet \mapsto \emptyset$ (empty Dyck word)

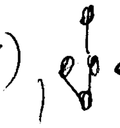
$T =$ 

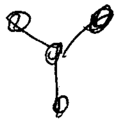
$\longmapsto (1, f(T_1), -1, 1, f(T_2), -1, \dots, 1, f(T_k), -1)$ $= f(T)$.

☒ (Exercise: Prove formally, that this f is the desired bijection)

Corollary $|\mathcal{D}_n| = |\mathcal{T}_{n+1}| = c_{n+1}$, where $\mathcal{D}_n$ is the

④ set of Dyck words of size n. | the five D. words of size 3 are:

 $\leftrightarrow (1,-1,1,-1,1,-1)$,  $\leftrightarrow (1,1,1,-1,-1,-1)$,  $\leftrightarrow (1,-1,1,1,-1,-1)$;

 $\leftrightarrow (1,1,-1,-1,1,-1)$,  $\leftrightarrow (1,1,-1,1,-1,-1)$.

We count $\mathcal{D}_n$. $\mathcal{E}_n := \{$all $2n$-tuples with n 1s and n -1s$\}$.

$\mathcal{F}_n := \{$ -||- $(n+1)$ 1s and $(n-1)$ -1s$\}$.

Clearly, $|\mathcal{E}_n| = \binom{2n}{n}$, $|\mathcal{F}_n| = \binom{2n}{n+1} = \binom{2n}{n-1}$. Also, $\mathcal{D}_n \subseteq \mathcal{E}_n$. **Proposition** $\forall n\ \exists$ a bijection $f_n : \mathcal{E}_n \setminus \mathcal{D}_n \to \mathcal{F}_n$.

Proof. $\mathcal{E}_n \setminus \mathcal{D}_n \ni D = (d_1, d_2, \dots, d_{2n}) \overset{f_n}{\longmapsto} (-d_1, -d_2, \dots, -d_j, d_{j+1}, \dots, d_{2n}) \in \mathcal{F}_n$ where j is minimum s.t. $d_1 + d_2 + \dots + d_j = -1$. The inverse g_n to f_n flips signs in the shortest initial interval in $(d_1, d_2, \dots, d_{2n}) \in \mathcal{F}_n$ with $d_1 + \dots + d_j = 1$. So f_n is a bijection. ☒

Thus

$$\binom{2n}{n-1} = |\mathcal{F}_n| = |\mathcal{E}_n| - |\mathcal{D}_n| \text{ and } \underset{C_{n+1}}{|\mathcal{D}_n|} = \binom{2n}{n} - \binom{2n}{n-1} =$$

$$= \binom{2n}{n} - |\mathcal{D}_n| \quad = \frac{(2n)!\left(1 - \frac{n}{n+1}\right)}{n!\, n!} =$$

$$= \frac{1}{n+1}\binom{2n}{n}.$$

We proved that $C_n = \frac{1}{n}\binom{2n-2}{n-1}$.

⑤ Another 6 combinatorial families counted by the Catalan numbers. Let $\pi = (a_1, a_2, \dots, a_m) \in S_m$ and $\rho = (b_1, b_2, \dots, b_n) \in S_n$ be an m-permutation and an n-perm., resp. We define $\pi < \rho$ if $\exists$ indices $1 \le i_1 < i_2 < \dots < i_m$ s.t. $b_{i_j} < b_{i_k} \Leftrightarrow a_j < a_k$ $(\forall j, k)$. Then

Theorem $\forall \pi \in S_3$:

$$\#\{\rho \in S_n \mid \rho \not> \pi\} = C_{n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

For example, the # of n-perm. with no 3-term increasing subsequence is $C_{n+1} = \frac{1}{n+1}\binom{2n}{n}$.