L12, May 22, 2020 | We continue with the combin. aspect of the TMM formula. For a walk $\Gamma = (e_1, e_2, \dots, e_n)$ we define its weight $w(\Gamma) := \prod_{i=1}^{n} w(e_i)$. We further assume that D is finite, i.e. V and E are finite sets. We then define the $p \times p$ matrix $A \in R^{p\times p}$ by ($V = \{v_1, v_2, \dots, v_p\}$)

$$A_{i,j} := \sum_{e \in E,\ \dots} w(e)$$

where we sum over all edges e satisfying that $in(e) = v_i$, $fi(e) = v_j$ (*) (A is so called adjacency matrix of D and w). We define more generally, for $n \in \mathbb{N}_0$ and $i, j \in [p]$,

$$A_{i,j}(n) := \sum_{\Gamma: \dots} w(\Gamma)$$

with sum over all walks (in D) with length n from v_i to v_j ($= 0_R$ if no such Γ exists). Thus $A(1) = A$ and

$$A(0)_{i,j} = \delta_{i,j} = \begin{cases} 1_R & \dots i = j \\ 0_R & \dots i \neq j \end{cases} = I \text{ (} p \times p \text{ identity matrix in } R^{p\times p}\text{)}$$

Proposition $\forall i, j \in [p]\ \forall n \in \mathbb{N}_0$: $A_{i,j}(n) = (A^n)_{i,j}$

Proof.

$$(A^n)_{i,j} = \sum_{i_1, i_2, \dots, i_{n-1} = 1}^{p} \underbrace{A_{i,i_1} A_{i_1,i_2} \cdots A_{i_{n-1},j}}_{= \sum_{\Gamma} w(\Gamma)} = A_{i,j}(n).$$

where Γ is a walk from v_i to v_j over the vertices $v_{i_1}, v_{i_2}, \dots, v_{i_{n-1}}$, and $= 0_R$ if there is no such Γ. ☒

(*) $A_{i,j} := 0_R$ if there is no such edge.

$\boxed{\text{Theorem (TMM, combinatorial aspect)}}$

Suppose $D = (V, E, \varphi)$ is a finite digraph, $V = \{v_1, v_2, \dots, v_p\}$, $w : E \to R$ is a weight function, and $A \in R^{p \times p}$ is the adjacency matrix (and $A(n) \in R^{p \times p}$, $n \in \mathbb{N}_0$, are the matrices of weights of length n walks in D). Then for $i, j \in [p]$ we have that

$$\sum_{k=0}^{\infty} A_{i,j}(k) x^k = \frac{(-1)^{i+j} \det\big((I - xA)[j, i]\big)}{\det(I - xA)}$$

Proof. Follows at once from the previous Proposition and Theorem (TMM, algebraic part). $\boxtimes$

In particular the GF is rational.

We conclude this course with the LIF, or Lagrange inversion formula, but without proofs (no time).

If $f, g \in \mathbb{C}[[x]]$ are two formal power series with $g(0) = 0$, we define their composition $f(g(x)) = f \circ g \in \mathbb{C}[[x]]$ as follows. If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=1}^{\infty} b_n x^n$, then

$$f(g(x)) = f\circ g = \sum_{n=0}^{\infty} a_n\, g(x)^n = \sum_{n=0}^{\infty} a_n (b_1 x + b_2 x^2 + \dots)^n =$$

$$=: \sum_{n=0}^{\infty} c_n x^n$$ where $c_0 = a_0$ and for $n \geq 1$,

$$c_n = \sum_{j=1}^{n} a_j \sum_{\substack{m_1,\dots,m_j \in \mathbb{N} \\ \cancel{m_1+m_2+\dots+m_j=n} \\ \cancel{m_1+2m_2+3m_3+\dots+jm_j} \\ m_1+m_2+\dots+m_j=n}} \cancel{\binom{n}{m_1, m_2, \dots, m_j}}\, b_{m_1} b_{m_2} \cdots b_{m_j}.$$

One can show that this composition operation $\circ$ is associative. If $g(0) \neq 0$, $f\circ g$ is not defined. It is clear that $g(x) = x$ is the both-sided neutral element to $\circ$, $x\circ g(x) = g(x)\circ x$ for every $g \in \mathbb{C}[[x]]$ with $g(0) = 0$. Let

$$\mathbb{C}[[x]]_1 := \{ f \in \mathbb{C}[[x]] \mid f(x) = a_0 + a_1 x + a_2 x^2 + \dots \text{ s.t. } a_0 = 0 \ \&\ a_1 \neq 0 \}.$$ This set is clearly closed to the operation $\circ$ and so

$$\mathbb{C}[[x]]_1 = (\mathbb{C}[[x]]_1, x, \circ)$$ is a <u>monoid</u> — $\circ$ is associative and x is the neutral element.

$\boxed{\text{Proposition}}$ If $M = (M, 1_M, \cdot)$ is a monoid such that $\forall a \in M\ \exists b \in M : ba = 1_M$ — we say that a has a left inverse $a^{-1} := b$ then

$\forall a \in M$ the element a^{-1} is also ~~the~~ a right inverse of a, $a a^{-1} = 1_M$. Also, this both-sided inverse is unique for every $a \in M$.

Proof: Let $a \in M$ be arbitrary and let $b := a a^{-1}$. Then, by associativity and presence of 1_M,

$$b^2 = bb = (a a^{-1})(a a^{-1}) = a(a^{-1} a) a^{-1} =$$
$$= a 1_M a^{-1} = a a^{-1} = b.$$

Thus b is so called idempotent. But then also (since b has a left inverse b^{-1})

$$1_M = b^{-1} b = b^{-1}(bb) = (b^{-1} b) b = 1_M b = b = a a^{-1}.$$

Thus a^{-1} is also right inverse of a. Suppose that a^{-1} and a' are two left inverses of a. Then

$$a^{-1} = 1_M a^{-1} = (a'a) a^{-1} = \cancel{a'a a}^{-1} = a'(a a^{-1}) = a' 1_M = a'.$$

Thus a^{-1} is the unique left inverse of a. Similarly a^{-1} is unique as a right inverse. $\boxtimes$

We call a^{-1} then simply the inverse of a.

The Proposition applies to the monoid $\mathbb{C}[[x]]_1$ because one can show (try it as an exercise) that $\forall f \in \mathbb{C}[[x]]_1$ has a left inverse. In fact, no $f \in \mathbb{C}[[x]] \setminus \mathbb{C}[[x]]_1$ has a left inverse (or right). (5)

We denote the inverse by $\boxed{f^{\langle -1\rangle}}$

For example, the GF of Catalan numbers $C = C(x) = \sum_{n=1}^{\infty} \frac{1}{n}\binom{2n-2}{n-1}x^n$ satisfies the quadratic equation $C - C^2 = x$, thus $C(x)^{\langle -1\rangle} = x - x^2$. By the proposition also $C(x-x^2) = x$.

(*) easy to see.

Theorem (the LIF) Let $f(x) = x + a_2x^2 + \dots \in \mathbb{C}[[x]]$ then $\forall n \in \mathbb{N}$:

$$[x^n]\, f(x)^{\langle -1\rangle} = \frac{1}{n}[x^{n-1}]\left(\frac{x}{f(x)}\right)^n.$$

For example, since $C(x) = (x-x^2)^{\langle -1\rangle}$, by we have that

$$c_n = [x^n]\, C(x) = [x^n]\,(x-x^2)^{\langle -1\rangle} =$$

$$= \frac{1}{n}[x^{n-1}]\left(\frac{x}{x-x^2}\right)^n = \frac{1}{n}[x^{n-1}]\left(\frac{1}{1-x}\right)^n =$$

$$= \frac{1}{n}[x^{n-1}](1-x)^{-n} = \frac{1}{n}(-1)^{n-1}\binom{-n}{n-1} =$$

$$= \frac{1}{n}\frac{(-n)(-n-1)\dots(-2n+2)}{(n-1)!}(-1)^{n-1} = \frac{1}{n}\frac{(2n-2)\dots(n+1)n}{(n-1)!}$$

$= \frac{1}{n}\binom{2n-2}{n-1}$. We end where we started 11lecenturies ago, at the Catalan numbers!

Thanks you for your attention and patience!