L 11 May 15, 2020

The rings $\overset{R[[x]]^{n\times n}}{U}$ and $\overset{R^{n\times n}[[x]]}{V}$ are non-commutative and have zero divisors (non-zero elements with zero product). Let us denote, for $k\in\mathbb{N}_0$ and x a formal variable, by $[x^k](a_0+a_1x+a_2x^2+\dots):=$ $:=a_k$, the coeff. of x^k in the expansion following after the symbol $[x^k]$.

Proposition (an isomorphism)

Let R be a ring (commutative and with 1_R) and $U=R[[x]]^{n\times n}$ and $V=R^{n\times n}[[x]]$ be the ring of $n\times n$ matrices whose entries are (formal power series) $a_0+a_1x+a_2x^2+\dots$ with $a_i\in R$ and the ring of formal power series $A_0+A_1x+A_2x^2+\dots$ with $A_i\in R^{n\times n}$ ($n\times n$ matrices with entries in R), respectively. The maps

$$\kappa: U\to V,\ M\mapsto \sum_{k=0}^{\infty}\left([x^k]M_{i,j}\right)_{i,j=1}^{n}x^k$$

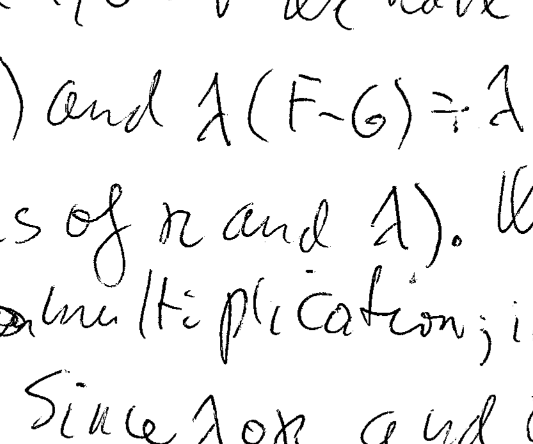
and

$$\lambda: V\to U,\ \sum_{k=0}^{\infty}M(k)x^k\longmapsto\left(\sum_{k=0}^{\infty}M(k)_{i,j}x^k\right)_{i,j=1}^{n}$$

are mutually inverse homomorphisms of

rings. Thus $\varkappa$ and λ are isomorphisms of the rings U and V.

Proof. It is clear that $\varkappa$ and λ send $1_U \mapsto 1_V$ and $1_V \mapsto 1_U$, respectively. Also, $\varkappa$ and λ preserve subtraction, i.e. for any $A, B \in U$ and $F, G \in V$ we have that $\varkappa(A-B) = \varkappa(A) - \varkappa(B)$ and $\lambda(F-G) = \lambda(F) - \lambda(G)$ (check the definitions of $\varkappa$ and λ). We show that they also preserve ~~the~~ multiplication; it is only a challenge in notation. Since $\lambda \circ \varkappa$ and $\varkappa \circ \lambda$ are identical maps, $\varkappa$ and λ are mutually inverse bijections and it suffices only to check multiplicativity of, say, the map λ:



Let $F, G \in V$ be given formal power series with matrix coeff-s, $F = \sum_{q=0}^{\infty} M(q)x^q$ and $G = \sum_{q=0}^{\infty} N(q)x^q$, where $M(q), N(q) \in R^{n\times n}$. Then $\lambda(FG) \overset{(p.m.)}{=}$

$$\overset{(p.m.)}{=} \lambda\left(\sum_{q=0}^{\infty}\left(\sum_{l=0}^{q} M(l)N(q-l)\right)x^q\right) =$$

$= \lambda\left(\sum_{q=0}^{\infty} P(q)x^q\right)$ where the matrices $P(q)$ ~~are~~ have entries ~~given by~~ $(i,j\in[n])$, $P(q)_{i,j} \overset{(m.m.)}{=} \sum_{l=0}^{q}\sum_{i_1=1}^{n} M(l)_{i,i_1} N(q-l)_{i_1,j}$.

Therefore

$$\lambda(FG) \overset{\text{by def. of } \lambda}{=} \left(\sum_{q=0}^{\infty} P(q)_{i,j}x^q\right)_{i,j=1}^{n}$$

$$\overset{(m.m.)}{\underset{(p.m.)}{=}} \left(\sum_{q=0}^{\infty}\left(\sum_{l=0}^{q}\sum_{i_1=1}^{n} M(l)_{i,i_1}N(q-l)_{i_1,j}\right)x^q\right)_{i,j=1}^{n}.$$

On the other hand, $\lambda(F)\lambda(G) \overset{\text{by def. of } \lambda}{=}$

$$\overset{(\lambda)}{=} \left(\sum_{q=0}^{\infty} M(q)_{i,j}x^q\right)_{i,j=1}^{n}\left(\sum_{q=0}^{\infty} N(q)_{i,j}x^q\right)_{i,j=1}^{n}$$

$$\overset{(m.m.)}{=} \left(\sum_{i_1=1}^{n}\left(\sum_{q=0}^{\infty} M(q)_{i,i_1}x^q\right)\left(\sum_{q=0}^{\infty} N(q)_{i_1,j}x^q\right)\right)_{i,j=1}^{n}$$

$$\overset{(p.m.)}{=} \left(\sum_{i_1=1}^{n}\sum_{q=0}^{\infty}\left(\sum_{l=0}^{q} M(l)_{i,i_1}N(q-l)_{i_1,j}\right)x^q\right)_{i,j=1}^{n}.$$

Above (λ), (m.m.) and (p.m.) mean that the def. of λ, matrix multiplication and formal power series multiplication was applied, respectively.

In the last $\lfloor \dots \rfloor$ we change order of summation, ~~and~~ move the finite outer sum $\sum_{i_1=1}^{n} \dots$ completely inside and get the previous $\lfloor \dots \rfloor$. Hence $\lambda(F)\lambda(G) = \lambda(FG)$. Why could we move the $\sum_{i_1=1}^{n} \dots$ completely inside? The first exchange $\sum_{i_1=1}^{n} \sum_{k=0}^{\infty} \to \sum_{k=0}^{\infty} \sum_{i_1=1}^{n}$ is in fact an application of the ~~coefficient-wise~~ addi- definition of addition of f. power series as coefficient-wise addition. The next exchange $\sum_{i_1=1}^{n} \sum_{l=0}^{k} \to \sum_{l=0}^{k} \sum_{i_1=1}^{n}$ is the combinatorial double counting trick, which we can here state abstractly and generally as follows.

$\boxed{\text{Lemma (double counting)}}$ Let $S = (S, +)$ be an associative and commutative groupoid (i.e. $+$ is a binary operation on S which is a. and c.). If $a_{i,j} \in S$ for $i \in [m]$ and $j \in [n]$ ($m, n \in \mathbb{N}$) then

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j} = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{i,j}.$$

Proof: An exercise for you. (*) ☒

This finishes the proof of the Proposition ☒

Let us revisit in the perspective of the isomorphisms $\varkappa$ and λ the derivation of the TMM formula. For the given $A \in R^{n \times n}$ we take the matrix $M := \left(\sum_{q=0}^{\infty} (A^q)_{i,j} x^q \right)_{i,j=1}^{n} \in U$. We apply $\varkappa$ and get the f. power series $F := \varkappa(M) = \sum_{q=0}^{\infty} A^q x^q \in V$. In V,

$$\left(\sum_{q=0}^{\infty} A^q x^q\right)\left(I \cdot x^0 - Ax\right) = \sum_{q=0}^{\infty} A^q x^q - \sum_{q=0}^{\infty} A^{q+1} x^q =$$

$= I x^0$, so in V we have that $F = (Ix^0 - Ax)^{-1}$. (Since V is a ring of f. power series with non-commutative coeff-s, one has to check carefully applicability of the geometric series formula.) Isomorphisms of rings preserve inverse elements, by applying λ we therefore get that $\lambda(F) \in U$ actually is $\lambda(F) = (I - xA)^{-1}$! Applying the (Cramer's (*)) formula for inverse matrix we obtain that

$$= \left(\frac{(-1)^{i+j} \det((I - xA)[j|i])}{\det(I - xA)} \right)_{i,j=1}^{n}$$

$N :=$

Hence

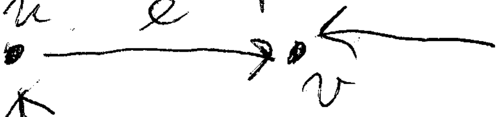
(*) Formalized proof of the d-counting lemma could be of interest.

$N = \lambda(F) = \lambda(\varkappa(M)) = (\lambda \circ \varkappa)(M) = M$.

Thus we get the TOMM formula that $\forall\, i,j \in [n]$,

$$\boxed{M_{i,j} = N_{i,j}}.$$

Let us now finally turn to the combinatorial side of the TOMM formula. We have a directed multigraph (shortly $\boxed{\text{digraph}}$) $D = (V, E, \varphi)$ where $V = \{v_1, v_2, \dots, v_p\}$ are the vertices, E are the edges and φ is a map $\varphi: E \to V \times V$. If $\varphi(e) = (u, v)$,

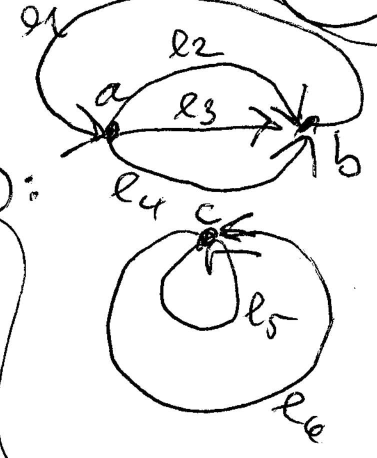


initial vertex of e

final vertex of e, $u =: in(e)$, $v =: fi(e)$

$u = v$:  is a loop. So D may look like:

$E = \{e_1, \dots, e_6\}$, $V = \{a, b, c\}$

D: 

A walk Γ in D from $u \in V$ to $v \in V$, of length n, is a sequence $e_1 e_2 \dots e_n$ of n edges such that $in(e_1) = u$, $fi(e_n) = v$ and $fi(e_i) = in(e_{i+1})$ for $i = 1, 2, \dots, n-1$. Γ is closed if $u = v$. We consider a weight function $w: E \to R$, where R is any ring (comm., with 1_R).

To be continued in the last lecture