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Deterministic Approach to the Kinetic Theory of Gases

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Abstract In the so-called Bernoulli model of the kinetic theory of gases, where (1) the particles are dimensionless points, (2) they are contained in a cube container, (3) no attractive or exterior forces are acting on them, (4) there is no collision between the particles, (5) the collision against the walls of the container are according to the law of elastic reflection, we deduce from Newtonian mechanics two local probabilistic laws: a Poisson limit law and a central limit theorem. We also prove some global law of large numbers, justifying that "density" and "pressure" are constant. Finally, as a byproduct of our research, we prove the surprising super-uniformity of the typical billiard path in a square.

Keywords Typical billiard path in a square and in a cube · Elastic reflection · Large billiard systems · Poisson limit law · Fourier analysis · Parseval's formula

1 Time-evolution in the Local Case: Poisson Limit Theorem

1.1 Where Does Randomness Come from?

This paper provides rigorous mathematical proofs to support the postulate in statistical mechanics that the particles are represented by independent random variables. I recall that the kinetic theory of gases describes gas as an accumulation of a very large (but finite) number N of rapidly moving tiny particles (N is at the order of 10^{20} per cm^3, the average speed is roughly around 10^3 meter per second at room temperature, depending on the gas, and the size of the particles is about 10^{-9}–10^{-10} meter). The particles (= molecules) are colliding with one another and against the wall of the container. Hence, if for the time-point $t = 0$ we know the space coordinates

$$(x_j(0), y_j(0), z_j(0)), \quad j = 1, 2, \ldots, N \tag{1.1}$$

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and the velocities

$$(\dot{x}_j(0), \dot{y}_j(0), \dot{z}_j(0)), \quad j = 1, 2, \ldots, N \tag{1.2}$$

of the particles (we call (1.1) and (1.2) the initial condition), the state of the system is *theoretically* determined for the entire future $t > 0$ too. Theoretically yes, but practically no: an effective determination of even the simplest properties of gas is completely hopeless to achieve in that way. Indeed, in order to compute the time evolution of the system of N particles, we would have to deal with $6N$ equations in $6N$ variables, which is of course a totally unrealistic task if N is in the range of 10^{20}.

It is the general view among physicists, therefore, that the basic properties of gas cannot be deduced from the principles of classical mechanics alone, and this impossibility was the basis for a probabilistic treatment called "statistical mechanics". Statistical mechanics is based on (often implicit) postulates involving non-Newtonian concepts such as *probability* and *statistical independence*—the later usually combined with uniform distribution. The physicists prefer to call it "equal a priori probabilities in the phase space"; see any textbook, e.g., Tolman [18], or Thompson [17], or Uhlenbeck–Ford [19].

As an illustration, consider the famous Maxwell–Boltzmann energy law

$$\text{Probability}(\text{energy} = E_j) = \frac{e^{-\beta E_j}}{\sum_i e^{-\beta E_i}}$$

(where $1/\beta = kT$, k is the Boltzmann's constant and T is the temperature), which is generally considered the single most important law in statistical mechanics. Every known mathematical "proof" of the Maxwell–Boltzmann energy law is based on the postulate of equal a priori probabilities in the phase space.

How can we justify the Equiprobability Postulate? How does probability enter Classical Mechanics? Unfortunately, the task of finding a rigorous mathematical foundation for Statistical Mechanics remains largely unsolved. The objective of this paper is exactly to give a new insight to this long-standing open problem.

We have to admit, however, that the lack of rigorous mathematical foundations is not such a big headache for the physicists: the majority of them are pragmatists anyway. They are perfectly satisfied with the fact that Statistical Mechanics works: it can correctly predict the outcomes of (most of) the experiments. Agreement with experiment is the best substitute for a rigorous mathematical proof of the Equiprobability Postulate.

Physicists say: "try this; if it works (with reasonable level of accuracy) that will justify the postulate". In this paper I represent the viewpoint of a mathematician. With all due respect (and admiration!) to the physicists, a mathematician by training is obliged to point out the characteristic fallacy: "inductive experience that the postulate works is not a rigorous mathematical proof".

From Physics to Mathematics: Probability Theory What the physicists call *equal a priori probabilities in the phase space* is nothing else than the mathematical term (*statistical*) *independence with uniformly distributed components*. In other words, the simplest rigorous mathematical model in statistical mechanics describes the ideal gas in terms of independent and uniformly distributed random variables. More precisely, the physical system of an ideal gas of N particles in a cube container—say, the unit cube $[0, 1]^3$—is represented by N mutually independent random variables $X_1, X_2, \ldots, X_N$, where each X_j is uniformly distributed in $[0, 1]^3$, meaning that for any measurable subset $A \subset [0, 1]^3$, $\Pr[X_j \in A] = \text{volume}(A)$.

Here is a simple but important question that we can easily answer in this probabilistic model. What is the distribution of the number of particles of an ideal gas lying in a given fixed domain $A \subset [0,1]^3$ of very small volume $\mathrm{vol}(A) = \frac{1}{N}$?

Let X_A denote the number of particles lying in A; it is a random variable. The expected value of X_A is clearly 1:

$$\mathbf{E}X_A = N \cdot \frac{1}{N} = 1,$$

and for any integer $0 \le k \le N$, we have the probability

$$\Pr[X_A = k] = \binom{N}{k} \frac{1}{N^k} \left(1 - \frac{1}{N}\right)^{N-k}. \tag{1.3}$$

(Of course, (1.3) is 0 if $k > N$.) If k is fixed and $N \to \infty$, then we have the well-known limit

$$\Pr[X_A = k] = \frac{1}{k!}\left(1 - \frac{1}{N}\right)^N \cdot \frac{N(N-1)\cdots(N-k+1)}{N^k} \to \frac{1}{k!}e^{-1}, \tag{1.4}$$

which is a special case of the Poisson Limit Theorem.

If we switch the (mathematical) expectation from 1 to an arbitrary positive constant $\lambda > 0$, that is, $\mathrm{vol}(A) = \frac{\lambda}{N}$, then

$$\lim_{N\to\infty} \Pr[X_A = k] = \frac{\lambda^k}{k!}e^{-\lambda}, \tag{1.5}$$

which is the general case of the Poisson Limit Theorem. (Note that for the Poisson distribution with parameter $\lambda > 0$ (see (1.5)) the expectation and the variance are both equal to λ.)

Next let $A_1, A_2, \ldots, A_r$ be a finite sequence of disjoint measurable subsets of the unit cube $[0,1]^3$, and assume that $\mathrm{vol}(A_i) = \lambda_i/N$, $i = 1, 2, \ldots, r$. We study the distribution of the vector-valued random variable $(X_{A_1}, X_{A_2}, \ldots, X_{A_r})$, where X_{A_i} denotes the number of particles lying in A_i (I recall that the N particles are represented by N mutually independent random variables $X_1, X_2, \ldots, X_N$, where each X_j is uniformly distributed in $[0,1]^3$). Let $k_1, k_2, \ldots, k_r$ be an arbitrary sequence of non-negative integers with $0 \le k_1 + k_2 + \cdots + k_r \le N$. We have

$$\begin{aligned}
&\Pr[(X_{A_1}, X_{A_2}, \ldots, X_{A_r}) = (k_1, k_2, \ldots, k_r)] \\
&= \binom{N}{k_1}\binom{N-k_1}{k_2}\cdots\binom{N-k_1-\cdots-k_{r-1}}{k_r}\left(\frac{\lambda_1}{N}\right)^{k_1}\left(\frac{\lambda_2}{N}\right)^{k_2}\cdots\left(\frac{\lambda_r}{N}\right)^{k_r} \\
&\quad \cdot \left(1 - \frac{\lambda_1 + \lambda_2 + \cdots + \lambda_r}{N}\right)^{N-k_1-\cdots-k_r}.
\end{aligned}$$

If $k_1, k_2, \ldots, k_r$ are fixed and $N \to \infty$, then we have the simple limit

$$\begin{aligned}
&\lim_{N\to\infty} \Pr[(X_{A_1}, X_{A_2}, \ldots, X_{A_r}) = (k_1, k_2, \ldots, k_r)] \\
&= \frac{\lambda_1^{k_1}}{k_1!}e^{-\lambda_1} \cdot \frac{\lambda_2^{k_2}}{k_2!}e^{-\lambda_2} \cdots \frac{\lambda_r^{k_r}}{k_r!}e^{-\lambda_r}.
\end{aligned} \tag{1.6}$$

Comparing (1.5) to (1.6), it is natural to call (1.6) the Poisson *product* formula.

Let's return to (1.3). If we switch from constant to $\lambda = \lambda(N)$ with $\lambda(N) \to \infty$ but $\lambda(N)/N \to 0$ as $N \to \infty$, then (1.3) remains true with λ/N instead of $1/N$, but the limit in (1.5) is replaced with the De Moivre–Laplace limit

$$\lim_{N\to\infty} \sum_{c_1\sqrt{\lambda} \le k-\lambda \le c_2\sqrt{\lambda}} \Pr[X_A = k] = \frac{1}{\sqrt{2\pi}}\int_{c_1}^{c_2} e^{-u^2/2}\,du \tag{1.7}$$

for any fixed real numbers $-\infty < c_1 < c_2 < \infty$. Notice that (1.7) is a special case of the Central Limit Theorem (or "normal law", or "bell curve law", or "Gaussian distribution law") in the special case of the asymmetric binomial distribution.

The Poisson Limit Theorem and the Central Limit Theorem are the two most important limit theorems; they are the trademarks of probability theory.

Now we leave the probabilistic model, and return to Newtonian mechanics. The objective of this paper is to show that, in the case of a very simple deterministic model, namely, where the following five properties hold:

(1) the particles are dimensionless points of equal mass,
(2) they are contained in a cube container,
(3) no attractive or exterior forces are acting on them,
(4) there is no collision between the particles,
(5) the collisions against the walls of the container are according to the law of elastic reflection (i.e., the angle of incidence equals the angle of reflection),

we can deduce from the fundamental principles of mechanics the two probabilistic laws described in (1.3)–(1.7). More precisely, we prove that the time-evolution of the deterministic model exhibits a local Poisson Limit Theorem and a local Central Limit Theorem; see Theorem 1 below.

Also, we will prove a global *law of large numbers* implying that "the density is constant"; see Theorems 2 and 3 in Sect. 3. (The fourth main result, Theorem 4 in Sect. 4, is about the "super-uniformity" of the typical billiard paths in a square or a rectangle. The surprising message is that, the "ugliness" of the measurable subset $A \subset [0,1]^2$ we test the uniformity with, is basically irrelevant!)

We may call the model described by (1)–(5) the "Bernoulli model", after Daniel Bernoulli who introduced a similar model around 1738. What we do in this paper is a *quantitative* theory of the Bernoulli model, providing explicit error terms. Instead of relying on ergodic theory—which is considered the traditional mathematical approach to rigorous statistical mechanics—our approach is built around the Kronecker–Weyl equidistribution theorem and the use of hard Fourier analysis.

The five properties (1)–(5) of our simple deterministic model ("Bernoulli model") can be restated in an illuminating way in terms of point-billiard: N non-interacting billiard balls—each represented by a point mass—move freely inside a cube container, each one along a straight line, until one hits the wall (i.e., one of the six faces of the cube). The reflection off the wall is elastic; after the reflection the point (= billiard ball) continues its linear motion with the new velocity (but the speed remains the same; we ignore friction, air resistance, etc.) until its hits the wall again, and so forth. The same applies for all N billiard balls (= points = "molecules").

The initial condition, i.e., the starting point of the billiard path and the initial direction, uniquely determine an infinite piecewise linear billiard path $\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t))$, $0 < t < \infty$ in the unit cube. For simplicity, consider first the billiard path in the unit square; the law of reflection implies that there are at most four different directions along the billiard path: the initial direction is preserved modulo $\pi/2$ (= the angle of the square), which is one-fourth of the whole angle 2π. If we switch from the unit square to the d-dimensional unit cube with any $d \geq 3$, then again the law of reflection implies that there are only a bounded number of different directions along the billiard path (of course the bound depends on the dimension d).

Let's return to the point-billiard in the unit cube. The vague term of "typical billiard path" can be made precise very easily: we just have to define a measure on the set of all initial conditions of the billiard paths. The initial condition consists of a starting point $\mathbf{y} \in [0,1)^3$ and an initial direction $\mathbf{u} \in S^2$ (here $\mathbf{u}$ is a 3-dimensional unit vector and S^2 is the unit sphere; note that the speed remains constant as the time passes). Therefore, the corresponding measure is simply the product of the 3-dimensional Lebesgue measure in the unit cube ("volume") and the normalized surface area on the unit sphere S^2.

This way a vague term such as "$1-\varepsilon$ part of all billiard paths" becomes perfectly precise. Similar argument works for a large system of N point-billiards (we take the product measure, which is the natural measure in the phase space).

1.2 The Trick of Unfolding

Next we explain the well-known trick of *unfolding* the billiard path inside the unit cube to a straight line in the entire 3-space. The idea is very simple and elegant: we keep reflecting the unit cube in the respective face (where the path hits the boundary) and unfold the piecewise linear billiard path ("broken line") to a straight line. We strongly recommend the reader to draw a picture in the plane, and see how the "broken" billiard path becomes a straight line via unfolding (of course in the plane the *cube* is replaced by the *square*, and the *face* is replaced by the *side*).

Two straight lines in the 3-space correspond to the same billiard path if and only if they differ by a translation through an integral vector where both coordinates are even, i.e., where the vector is from the lattice $2\mathbb{Z}+2\mathbb{Z}+2\mathbb{Z}$. In other words, the problem of the distribution of a billiard path in the unit cube is equivalent to the distribution of the corresponding torus-line in the $2\times 2\times 2$ cube.

As far as I know, the first appearance of the geometric trick of unfolding is in a paper of D. König and A. Szücs from 1913, and it became widely known after Hardy and Wright included it in their famous book on number theory [5]. König and Szücs used the trick of unfolding (combined with the Kronecker–Weyl theorem) to prove the following elegant property of the billiard path in a square: if the slope of the initial direction is rational, then the billiard path is periodic, and if the slope of the initial direction is irrational, then the billiard path is dense, and what is more, it is uniformly distributed in the unit square (see [5]). Notice that the analog statement for torus-lines is the famous Kronecker–Weyl equidistribution theorem (I will return to the Kronecker–Weyl theorem later in Sects. 2–4).

1.3 Time-evolution in the Deterministic Bernoulli Model: Theorem 1

In our simplistic Bernoulli model the particles don't collide with one another, so the speed v_k of the kth particle remains constant, and, as I said above, the velocity is also basically constant: the velocity (= time-derivative)

$$\dot{\mathbf{x}}_j(t) = (\dot{x}_{j,1}(t), \dot{x}_{j,2}(t), \dot{x}_{j,3}(t))$$

of the jth particle can have only a few different values as $0<t<\infty$ (due to the elastic collisions against the walls, a consequence of the right angles in the cube). We can say,

therefore, that, unlike the position, the velocity of a fixed particle (= billiard ball) does not "mix" as $0<t<\infty$. To make our deterministic model more realistic, we could easily assume that the speeds $v_1,\ldots,v_N$ of the N particles satisfy the Maxwellian distribution (= normal distribution). But because the proof is rather complicated and the notation is quite messy, for simplicity we decided to restrict ourselves to the special case where all speeds are equal:

$$v_1 = v_2 = \cdots = v_N = v \geq 1.$$

Theorem 1 *Assume that N non-interacting billiard balls, each represented by a point mass, move freely inside the unit cube $I^3=[0,1]^3$ such that the reflection off the wall (= side of the cube) is elastic. Let $\mathbf{x}_j(t)=(x_{j,1}(t), x_{j,2}(t), x_{j,3}(t))$ describe the trajectory of the jth billiard ball (= point) in the time interval $0\leq t\leq T$, where $\mathbf{x}_j(0)=\mathbf{y}_j$ is the initial position, $\dot{\mathbf{x}}_j(0)=v\cdot\mathbf{u}_j$ is the initial velocity and $v>0$ is the common speed ($\mathbf{u}_j$ is a unit vector, i.e., $\mathbf{u}_j\in S^2=$ unit sphere; $N\geq 1$, $T>1$ and $v>1$ are arbitrary, but the theorem becomes interesting only if N and vT are both large).*

Let $A\subset I^3$ be an arbitrary Lebesgue measurable subset of the unit cube with volume $\mathrm{vol}(A)=\lambda/N$ (the range of parameter $\lambda>0$ will be given in (1.9) below). Let $Y_A(t)$ denote the point-counting function:

$$Y_A(t) = \sum_{\substack{1\leq j\leq N:\\ \mathbf{x}_j(t)\in A}} 1.$$

Let

$$m = \min\left\{\frac{e^{\frac{1}{2}\sqrt{\log(vT)}}}{101}, \sqrt{N}\right\} \quad \text{and} \quad \varepsilon = \frac{1}{\sqrt{m}}, \tag{1.8}$$

where log *denotes the natural (i.e., base e) logarithm.*

Then for more than $1-\varepsilon$ part of the initial conditions

$$\omega = (\mathbf{y}_1,\ldots,\mathbf{y}_N,\mathbf{u}_1,\ldots,\mathbf{u}_N) \in (I^3)^N \times (S^2)^N = \Omega$$

(in the sense of the product measure on Ω), the distribution of the point-counting function

$$Y_A(\omega;t) = Y_A(\mathbf{y}_1,\ldots,\mathbf{y}_N,\mathbf{u}_1,\ldots,\mathbf{u}_N;t)$$

is very close to the Poisson distribution with parameter λ (assuming N and vT are both large) in the following quantitative sense: for every real number

$$0<\lambda\leq\frac{\log m}{8} \tag{1.9}$$

and every integer $k\geq 0$,

$$\left|\frac{1}{T}\mathrm{measure}\{0\leq t\leq T: Y_A(\omega;t)=k\} - \frac{\lambda^k}{k!}e^{-\lambda}\right| < \varepsilon. \tag{1.10}$$

Finally, we can generalize (1.10) to get the following analog of the product formula (1.6). Let $A_1, A_2,\ldots,A_r$ be an arbitrary finite sequence of disjoint measurable subsets of the unit cube $[0,1]^3$ with $\mathrm{vol}(A_i)=\lambda_i/N$, $i=1,2,\ldots,r$. Then for more than $1-\varepsilon$ part of the *initial conditions $\omega \in \Omega$, (in the sense of the product measure on Ω), the distribution of the point-counting function*

$$\left|\frac{1}{T}\text{measure}\{0 \le t \le T : (Y_{A_1}(\omega; t), \ldots, Y_{A_r}(\omega; t)) = (k_1, \ldots, k_r)\} - \frac{\lambda_1^{k_1}}{k_1!}e^{-\lambda_1} \cdots \frac{\lambda_r^{k_r}}{k_r!}e^{-\lambda_r}\right| < r \cdot \varepsilon \tag{1.11}$$

holds for all $r \ge 1$, all vectors $(k_1, \ldots, k_r)$ of non-negative integers, and all

$$0 < \lambda_i \le \frac{\log m}{8} \quad (1 \le i \le r).$$

Remarks (a) In statistical mechanics one usually studies the limit process, sometimes called thermodynamics, where the ratio *particle/volume* remains a fixed constant as $N \to \infty$. More precisely, we replace the unit cube with a large cube of volume N/λ (i.e., the side length is $(N/\lambda)^{1/3}$) with some fixed constant $\lambda > 0$, and consider the limit $N \to \infty$ (i.e., the number of particles tends to infinity). For every N, let $A = A(N)$ be an arbitrary measurable subset of volume$(A(N)) = 1$ (a subset of the large cube of volume N/λ), and we study the distribution of the number of particles in $A = A(N)$ during a long time-interval $0 < t < T$. Theorem 1 makes it possible to carry out the limit process *precisely*. Theorem 1 implies that, independently of the way we take the double limit $N \to \infty$ and $T \to \infty$ (i.e., the relative relation of N and T is totally irrelevant), the number of particles in $A = A(N)$ has a definite *limit distribution*, which is the Poisson distribution with parameter $\lambda > 0$. We emphasize that the *relevant* limit in statistical mechanics is when first T is fixed and $N \to \infty$, and then, in the second step, $T \to \infty$.

(b) It is remarkable that the "complexity" of the given subset $A \subset [0, 1]^3$ does *not* play any role in the theorem. Of course we cannot say anything nontrivial about *all* possible measurable $A \subset [0, 1]^3$ simultaneously (since the volume of a billiard path is zero). We can easily generalize, however, Theorem 1 for an arbitrary infinite sequence $A_1, A_2, A_3, \ldots$ of measurable subsets of the unit cube with $\text{vol}(A_i) = \lambda_i/N$. The only necessary modification in (1.10) is to insert a *weight factor* in the upper bound:

$$\left|\frac{1}{T}\text{measure}\{0 \le t \le T : Y_{A_i}(\omega; t) = k\} - \frac{\lambda_i^k}{k!}e^{-\lambda_i}\right| < i\varepsilon$$

for all $i = 1, 2, 3, \ldots$.

(c) Theorem 1 is about a single time-interval $0 \le t \le T$, where T is arbitrary but fixed. It is natural to ask what happens in the sequential case, that is, when we study the distribution of the point-counting function $Y_A(t)$ as t runs in $0 \le t \le T$ simultaneously for all $0 < T < T_0$, where T_0 is some large real number. It is not too difficult to prove such a sequential version of Theorem 1 by using a straightforward adaptation of the so-called *dyadic method*, originally developed for orthogonal series.

We can "sequentialize" Theorem 1 as follows. For every

$$2 < T < e^{N^{1/8}} \tag{1.12}$$

write

$$m(T) = \min\left\{\frac{e^{\frac{1}{2}\sqrt{\log(vT)}}}{101}, \sqrt{N}\right\} \quad \text{and} \quad \varepsilon(T) = \frac{1}{\sqrt{m(T)}},$$

then (say) for more than 99.99 percent of the initial conditions

$$\omega = (\mathbf{y}_1, \ldots, \mathbf{y}_N, \mathbf{u}_1, \ldots, \mathbf{u}_N) \in (I^3)^N \times (S^2)^N = \Omega$$

the distribution of the point-counting function

$$Y_A(\omega; t) = Y_A(\mathbf{y}_1, \ldots, \mathbf{y}_N, \mathbf{u}_1, \ldots, \mathbf{u}_N; t)$$

is very close to the Poisson distribution with parameter λ in the following sense:

$$\left|\frac{1}{T}\text{measure}\{0 \le t \le T : Y_A(\omega; t) = k\} - \frac{\lambda^k}{k!}e^{-\lambda}\right| < \varepsilon(T) \tag{1.13}$$

holds for every T in (1.12), for every real number $0 < \lambda \le \frac{\log m(T)}{8}$ and for every integer $k \ge 0$.

Note that in the kinetic theory of gases the number of particles is $N \approx 10^{24}$, so the upper bound for T in (1.12) is in the range of e^{10^3}, which is "effectively infinite".

The message of this sequential version of Theorem 1 is the following: as more and more time passes, the distribution of the point-counting function $Y_A(t)$ gets closer and closer to the Poisson distribution, and the speed of convergence is basically independent of the number of particles. Nevertheless, the number of particles is crucial in an indirect way: it gives a natural limitation to the Poisson approximation.

(d) We can give an "ergodic theorem type" interpretation of Theorem 1 in the sense of the equality

$$\text{space-average} = \text{time-average}.$$

Indeed, at the beginning $t = 0$, the initial positions $\mathbf{x}_j(0) = \mathbf{y}_j$, $1 \le j \le N$ of the N point-billiards are independent and uniformly distributed random variables (uniformly distributed in the unit cube $[0, 1]^3$). So the number of points $Y_A(\omega; 0)$ at the start $t = 0$ in a given (measurable) subset $A \subset [0, 1]^3$ of volume $\text{vol}(A) = \lambda/N$ (ω is the complete initial condition) has the binomial distribution (let $0 \le k \le N$):

$$\Pr[Y_A(\omega; 0) = k] = \binom{N}{k}\left(\frac{\lambda}{N}\right)^k \cdot \left(1 - \frac{\lambda}{N}\right)^{N-k} \to \frac{\lambda^k}{k!}e^{-\lambda}$$

that approximates the Poisson distribution with parameter $\lambda > 0$ as λ is fixed and $N \to \infty$. On the other hand, by (1.10) and (1.13), for a typical but *fixed* initial condition $\omega_0 \in \Omega$, the time evolution of the counting function $Y_A(\omega_0; t)$ approximates the same Poisson distribution as $0 \le t \le T \to \infty$. Therefore, we can roughly say that

$$\text{space-average} = \text{Poisson distribution} = \text{time-average},$$

which is indeed in the spirit of the ergodic theorem. There is, however, a fundamental difference between Theorem 1 and (say) Birkhoff's individual ergodic theorem. The ergodic theorem is a soft/qualitative result: it does not say anything about the speed of convergence; it does not give any error term. Theorem 1, on the other hand, is a hard/quantitative result:

My (complicated) estimation in (3.26) is certainly not optimal: I am convinced that in reality the system in the example above achieves uniformity in a fraction of a second.

The proof of Theorem 3, similarly to Theorem 2, is postponed to Sect. 8. Both proofs are based on a second moment argument.

4 Super-uniformity of the Typical Billiard Path

4.1 What is Super-uniformity?

This section is a detour: I discuss a surprising byproduct of my research in deterministic kinetic theory of gases. By using the basic technique of this paper—Fourier analysis—I show that the typical billiard paths in a square (or any rectangle) are extremely uniform far beyond "common sense". What most experts would consider a "common sense expectation" is the square-root size ("random") error; what we can prove is the much smaller *square-root logarithm*(!). Since square-root logarithm is "almost" constant, our upper bound is essentially independent of the complexity of the test set. We can say, therefore, that (roughly speaking) "the ugliness of the test set is irrelevant". Or we can say that the set of typical billiard paths represents the family of most uniformly distributed curves in the square. Theorem 4 below makes these vague statements precise.

More precisely, we study the trajectory of a single point-billiard, and for simplicity we restrict ourselves to the unit square (2-dimensional case). We want to compare the *actual time*—i.e., the time the point-billiard spends in a given (measurable) subset A of the unit square—to the *expected time*. The *expected time* is area(A) times the total time, which reflects "perfect uniformity" (we assume that the speed is one).

In view of the trick of *unfolding* the billiard path to a straight line in the plane (explained in Sect. 1), it suffices to deal with torus-lines (of course we shrink the corresponding 2×2 square to the unit square). Let $A \subset I^2 = [0,1)^2$ be an arbitrary Lebesgue measurable subset (via unfolding it corresponds to the union of four copies of the given test-set), and consider the torus-line $\mathbf{x}(t) = (x_1(t), x_2(t))$ (mod 1) where

$$x_1(t) = \alpha_1 t + y_1, \qquad x_2(t) = \alpha_2 t + y_2 \quad \text{and} \quad \alpha_1^2 + \alpha_2^2 = 1. \tag{4.1}$$

The second part of (4.1) means that the speed is one, so the length of the straight line segment $\mathbf{x}(t)$, $0 < t < T$ is clearly T, i.e., time = arc-length. The initial condition $(\mathbf{y}, (\alpha_1, \alpha_2))$ describes the starting point $\mathbf{y} \in [0,1)^2$ and the angle (by the point (α_1, α_2) on the unit circle) of the torus-line $\mathbf{x}(t)$. Let $A(T) = A(T; \mathbf{y}, (\alpha_1, \alpha_2))$ denote the total time the torus-line $\mathbf{x}(t)$ (defined in (4.1)) spends in subset A during the given time interval $0 < t < T$. That is,

$$A(T; \mathbf{y}, (\alpha_1, \alpha_2)) = \text{actual time}, \quad \text{and} \quad \text{area}(A) \cdot T = \text{expected time},$$

and we want to compare the two.

I begin with the continuous form of the Kronecker–Weyl equidistribution theorem.

Theorem (Continuous Kronecker–Weyl) *If the slope α_2/α_1 is irrational, then for every starting point $\mathbf{y} \in [0,1)^2$,*

$$\lim_{T\to\infty} \frac{A(T; \mathbf{y}, (\alpha_1, \alpha_2))}{T} = \text{area}(A) \tag{4.2}$$

for all Jordan measurable sets $A \subset [0,1)^2$.

Note that a set in a euclidean space is Jordan measurable if and only if the characteristic function χ_A of the set is Riemann integrable.

The billiard path (4.1) has 2-dimensional Lebesgue measure zero, so (4.2) *cannot* be true for all Lebesgue measurable sets $A \subset [0,1)^2$. However, by involving the ergodic theorem, we can formulate a result, similar to (4.2), which holds for every Lebesgue measurable set $A \subset [0,1)^2$. First we have to define a dynamical system: for every real t we define the map

$$\Phi_t : \mathbf{y} \to \mathbf{y} + t(\alpha_1, \alpha_2) \pmod 1. \tag{4.3}$$

Clearly Φ_t is a mapping of the unit square $\mathbf{y} \in [0,1)^2$ into itself, and Φ_t preserves the Lebesgue measure ("area"). Since $\Phi_t(\mathbf{x}(0)) = \mathbf{x}(t)$ (see (4.1)), it is customary to call Φ_t the "time-shift". The quadruplet

$$\left([0,1)^2, \mathcal{L}, \lambda, \Phi_t\right), \tag{4.4}$$

where $\mathcal{L}$ denotes the σ-algebra of all Lebesgue measurable sets $A \subset [0,1)^2$ and λ is the 2-dimensional Lebesgue measure, is an *ergodic* dynamical system *if* the slope α_2/α_1 is irrational. Note that ergodicity follows from (4.2).

Next we apply Birkhoff's ergodic theorem: it states that, given any Lebesgue measurable set $A \subset [0,1)^2$, for *almost every* starting point $\mathbf{y} \in [0,1)^2$,

$$\lim_{T\to\infty} \frac{A(T; \mathbf{y}, (\alpha_1, \alpha_2))}{T} = \text{area}(A) = \lambda(A). \tag{4.5}$$

Since the classes of Jordan and Lebesgue measurable sets are both very large, and contain arbitrarily "ugly" (= complicated) sets $A \subset [0,1)^2$, it is not too surprising that neither the continuous Kronecker–Weyl Theorem (4.2), nor the ergodic Theorem (4.5) can say anything about the speed of convergence. In both cases (4.2) and (4.5),

$$|A(T; \mathbf{y}, (\alpha_1, \alpha_2)) - \text{area}(A) \cdot T| = o(T), \tag{4.6}$$

but we know nothing beyond that.

However, if we replace "every irrational (= ergodic) slope" with "almost every slope", then we can upgrade the weak (4.6) to a shockingly strong upper bound for the discrepancy:

$$|A(T; \mathbf{y}, (\alpha_1, \alpha_2)) - \text{area}(A) \cdot T| = O\left(\sqrt{\log T}\right). \tag{4.7}$$

Theorem 4 below is exactly an explicit/precise version of (4.7).

Throughout $\log x$ and $\log_2 x$ stand for the natural (i.e., base e) and the binary (i.e., base 2) logarithms (I don't use ln at all).

Theorem 4 *Let A be an arbitrary Lebesgue measurable subset of the unit square $[0,1)^2$ with two-dimensional Lebesgue measure* area(A), *and let $T > 100$ be an arbitrarily large (but fixed) real number. Let $\mathbf{x}(t) = (x_1(t), x_2(t))$, $0 \le t \le T$ be a billiard path of length T (= time) in the unit square, and let $A(T)$ denote the time the billiard path spends in subset A:*

$$A(T) = \text{measure}\{t \in [0,T] : \mathbf{x}(t) \in A\}.$$

Let $0 < \varepsilon < 1/2$ be arbitrary. Then for $1 - \varepsilon$ part of all billiard paths of length T in the square,

$$|A(T) - T \cdot \text{area}(A)| < \frac{10}{\varepsilon}\sqrt{\text{area}(A)(1 - \text{area}(A))} \cdot \sqrt{\log_2 T} \cdot \log_2 \log_2 T. \tag{4.8}$$

6 Proof of Theorem 4

In view of the trick of *unfolding* the billiard path to a straight line in the plane (explained in Sect. 1), it suffices to deal with torus-lines (of course we shrink the corresponding 2×2 square to the unit square). Let $A \subset I^2 = [0,1)^2$ be an arbitrary Lebesgue measurable subset (via unfolding it corresponds to the union of four copies of the given subset A in Theorem 4), and consider the Fourier series of the 0-1 valued characteristic function χ_A of the set A:

$$\chi_A(\mathbf{u}) = \sum_{\mathbf{r}\in\mathbb{Z}^2} a_{\mathbf{r}} e^{2\pi i \mathbf{r}\cdot\mathbf{u}} \quad \text{with } a_{\mathbf{r}} = \int_A e^{-2\pi i \mathbf{r}\cdot\mathbf{y}}\, d\mathbf{y}, \tag{6.1}$$

where $\mathbf{r}\cdot\mathbf{u} = r_1u_1 + r_2u_2$ denotes the standard inner product of vectors. Clearly $a_0 = \text{area}(A)$ (= the 2-dimensional Lebesgue measure of A), and by Parseval's formula,

$$\sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ \mathbf{r}\neq 0}} |a_{\mathbf{r}}|^2 = \text{area}(A) - \text{area}^2(A). \tag{6.2}$$

Consider the torus-line $\mathbf{x}(t) = (x_1(t), x_2(t))$ (mod 1) where

$$x_1(t) = \alpha_1 t + y_1, \qquad x_2(t) = \alpha_2 t + y_2 \quad \text{and} \quad \alpha_1^2 + \alpha_2^2 = 1. \tag{6.3}$$

The length of the straight line segment $\mathbf{x}(t)$, $0 < t < T$ is clearly T, i.e., time = arc-length. The pair $(\mathbf{y}, (\alpha_1, \alpha_2))$ describes the starting point $\mathbf{y} \in [0,1)^2$ and the angle (by the point (α_1, α_2) on the unit circle) of the torus-line $\mathbf{x}(t)$. The total time $A(T) = A(T; \mathbf{y}, (\alpha_1, \alpha_2))$ that the torus-line $\mathbf{x}(t)$ (defined in (6.3)) spends in subset A during $0 < t < T$ equals

$$\begin{aligned} A(T) = A(T; \mathbf{y}, (\alpha_1,\alpha_2)) &= \text{measure}\{t \in [0,T] : \mathbf{x}(t) \in A \text{ (mod 1)}\} \\ &= \int_0^T \chi_A(\mathbf{x}(t))\, dt = \int_0^T \sum_{\mathbf{r}\in\mathbb{Z}^2} a_{\mathbf{r}} e^{2\pi i \mathbf{r}\cdot\mathbf{x}(t)}\, dt \\ &= \sum_{\mathbf{r}\in\mathbb{Z}^2} a_{\mathbf{r}} \int_0^T e^{2\pi i \mathbf{r}\cdot\mathbf{x}(t)}\, dt = \sum_{\mathbf{r}\in\mathbb{Z}^2} a_{\mathbf{r}} e^{2\pi i \mathbf{r}\cdot\mathbf{y}} \int_0^T e^{2\pi i(\alpha_1 r_1 + \alpha_2 r_2)t}\, dt \\ &= a_0 T + \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ \mathbf{r}\neq 0}} a_{\mathbf{r}} e^{2\pi i \mathbf{r}\cdot\mathbf{y}} \cdot \frac{e^{2\pi i(\alpha_1 r_1 + \alpha_2 r_2)T} - 1}{2\pi i(\alpha_1 r_1 + \alpha_2 r_2)}. \end{aligned} \tag{6.4}$$

Since $a_0 = \text{area}(A)$ (= Lebesgue measure of A), by (6.4) we have

$$\begin{aligned} \text{discrepancy} &= A(T; \mathbf{y}, (\alpha_1,\alpha_2)) - T \cdot \text{area}(A) \\ &= \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ \mathbf{r}\neq 0}} a_{\mathbf{r}} \cdot \frac{e^{2\pi i(\alpha_1 r_1 + \alpha_2 r_2)T} - 1}{2\pi i(\alpha_1 r_1 + \alpha_2 r_2)} \cdot e^{2\pi i \mathbf{r}\cdot\mathbf{y}}. \end{aligned} \tag{6.5}$$

Fix any point (α_1, α_2) on the unit circle $\alpha_1^2 + \alpha_2^2 = 1$, and run the starting point $\mathbf{y}$ through the unit square; then by Parseval's formula ($I^2 = [0,1]^2$)

$$\int_{I^2} (A(T; \mathbf{y}, (\alpha_1,\alpha_2)) - T \cdot \text{area}(A))^2\, d\mathbf{y}$$

$$= \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ \mathbf{r}\neq 0}} |a_{\mathbf{r}}|^2 \cdot \left| \frac{e^{2\pi i(\alpha_1 r_1 + \alpha_2 r_2)T} - 1}{2\pi(\alpha_1 r_1 + \alpha_2 r_2)} \right|^2. \tag{6.6}$$

Let's study the last factor in (6.6): we have the obvious upper bound

$$\left| \frac{e^{2\pi i(\alpha_1 r_1 + \alpha_2 r_2)T} - 1}{2\pi(\alpha_1 r_1 + \alpha_2 r_2)} \right| \le \min\left\{ \frac{1}{\pi|\alpha_1 r_1 + \alpha_2 r_2|}, T \right\}. \tag{6.7}$$

Key Definition Let $0 < \varepsilon < 1/2$; we say that a point (α_1, α_2) on the unit circle $\alpha_1^2 + \alpha_2^2 = 1$ is ε-bad if there exists an $\mathbf{r} \in \mathbb{Z}^2$ such that

$$|\alpha_1 r_1 + \alpha_2 r_2| \le \frac{\varepsilon}{40|\mathbf{r}| \cdot \log_2|\mathbf{r}| \cdot (\log_2\log_2|\mathbf{r}|)^2} \tag{6.8}$$

for some $|\mathbf{r}| \ge 8$ or

$$|\alpha_1 r_1 + \alpha_2 r_2| \le \frac{\varepsilon}{40|\mathbf{r}|} \tag{6.9}$$

for some $1 \le |\mathbf{r}| < 8$, where $|\mathbf{r}| = \sqrt{r_1^2 + r_2^2}$.

Note that the complicated denominator in (6.8) is motivated by the fact that the numerical series

$$\sum_{n=3}^{\infty} \frac{1}{n(\log n)^2} \tag{6.10}$$

is very close to the border of convergence-divergence: the slightly larger series

$$\sum_{n=3}^{\infty} \frac{1}{n \log n}$$

is already divergent, but (6.10) is still convergent (see (6.15) below; of course we could replace the exponent 2 in (6.10) with $1 + \varepsilon$, but the gain would be negligible).

Next I show that the set $\mathcal{B}$ of all ε-bad points (α_1, α_2) on the unit circle $\alpha_1^2 + \alpha_2^2 = 1$ forms a small minority: the measure of $\mathcal{B}$ is negligible compared to the circumference 2π of the unit circle. (Note in advance that at the end we will throw out all initial conditions having ε-bad angles.)

Lemma 6.1 *The set $\mathcal{B}$ of ε-bad points (see the Key Definition) is small in the sense that*

$$\frac{\text{measure}(\mathcal{B})}{2\pi} < \frac{\varepsilon}{2}. \tag{6.11}$$

Proof Notice that $\alpha_1 r_1 + \alpha_2 r_2$ is a dot product of two vectors, so the absolute value $|\alpha_1 r_1 + \alpha_2 r_2|$ equals $|\mathbf{r}|\sin\theta$, where θ is the angle between the unit vector (α_1, α_2) and the vector $(-r_2, r_1)$ perpendicular to $\mathbf{r} = (r_1, r_2)$. Therefore, given any $\mathbf{r} \in \mathbb{Z}^2$ with $|\mathbf{r}| \ge 8$, inequality (6.8) defines two short diametrically opposite arcs on the unit circle $\alpha_1^2 + \alpha_2^2 = 1$ with total arc-length

$$4\arcsin\left(\frac{\varepsilon}{40|\mathbf{r}|^2 \cdot \log_2|\mathbf{r}| \cdot (\log_2\log_2|\mathbf{r}|)^2} \right),$$

where of course arcsin is the inverse of sin. Similarly, given any $\mathbf{r} \in \mathbb{Z}^2$ with $1 \le |\mathbf{r}| < 8$, inequality (6.9) defines two short diametrically opposite arcs on the unit circle $\alpha_1^2 + \alpha_2^2 = 1$ with total arc-length

$$4 \arcsin\left(\frac{\varepsilon}{40|\mathbf{r}|^2}\right).$$

It follows that

$$\text{measure}(\mathcal{B}) < \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ 1\le|\mathbf{r}|<8}} 4\arcsin\left(\frac{\varepsilon}{40|\mathbf{r}|^2}\right) + \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ |\mathbf{r}|\ge 8}} 4\arcsin\left(\frac{\varepsilon}{40|\mathbf{r}|^2\cdot\log_2|\mathbf{r}|\cdot(\log_2\log_2|\mathbf{r}|)^2}\right). \tag{6.12}$$

By using the trivial inequality

$$\arcsin(x) < x + x^2 \quad \text{for } 0 < x < 1, \tag{6.13}$$

we can easily estimate the sums in (6.12). We begin with the auxiliary sum

$$\sum\nolimits_1 = \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ |\mathbf{r}|\ge 8}} \frac{1}{|\mathbf{r}|^2\cdot\log_2|\mathbf{r}|\cdot(\log_2\log_2|\mathbf{r}|)^2}. \tag{6.14}$$

We estimate (6.14) by applying a standard power-of-two decomposition:

$$\begin{aligned}\sum\nolimits_1 &= \sum_{k=3}^{\infty}\sum_{2^k\le|\mathbf{r}|<2^{k+1}} \frac{1}{|\mathbf{r}|^2\cdot\log_2|\mathbf{r}|\cdot(\log_2\log_2|\mathbf{r}|)^2}\\ &< \sum_{k=3}^{\infty}\pi 4^{k+1}\cdot\frac{1}{4^k\cdot k\cdot(\log_2 k)^2}\\ &= 4\pi\sum_{k=3}^{\infty}\frac{1}{k\cdot(\log_2 k)^2}.\end{aligned} \tag{6.15}$$

Note that in (6.15) we used the trivial fact that the number of lattice points in the annulus $2^k \le |\mathbf{r}| < 2^{k+1}$ is less than the area of the big circle $\pi\cdot 4^{k+1}$.

Returning to (6.15), we can estimate the infinite series with the corresponding definite integral:

$$\sum_{k=3}^{\infty}\frac{1}{k\cdot(\log_2 k)^2} < \int_2^{\infty}\frac{dx}{x(\log_2 x)^2} = \log 2,$$

and using this in (6.15), we have

$$\sum\nolimits_1 < 4\pi\log 2. \tag{6.16}$$

Similarly,

$$\begin{aligned}\sum\nolimits_2 &= \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ |\mathbf{r}|\ge 8}}\frac{1}{|\mathbf{r}|^4\cdot(\log_2|\mathbf{r}|)^2\cdot(\log_2\log_2|\mathbf{r}|)^4}\\ &= \sum_{k=3}^{\infty}\sum_{2^k\le|\mathbf{r}|<2^{k+1}}\frac{1}{|\mathbf{r}|^4\cdot(\log_2|\mathbf{r}|)^2\cdot(\log_2\log_2|\mathbf{r}|)^4}\\ &< \sum_{k=3}^{\infty}\pi 4^{k+1}\cdot\frac{1}{16^k\cdot k^2\cdot(\log_2 k)^4} < \frac{\pi}{100}.\end{aligned} \tag{6.17}$$

We also need the simple numerical facts

$$\sum\nolimits_4 = \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ 1\le|\mathbf{r}|<8}}\frac{1}{|\mathbf{r}|^4} < \sum\nolimits_3 = \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ 1\le|\mathbf{r}|<8}}\frac{1}{|\mathbf{r}|^2} < 6\pi. \tag{6.18}$$

Combining (6.12)–(6.18), we have

$$\frac{\text{measure}(\mathcal{B})}{2\pi} < \frac{\varepsilon}{20\pi}\sum\nolimits_1 + \frac{\varepsilon^2}{800\pi}\sum\nolimits_2 + \frac{\varepsilon}{20\pi}\sum\nolimits_3 + \frac{\varepsilon^2}{800\pi}\sum\nolimits_4 < \frac{\varepsilon}{2},$$

completing the proof of Lemma 6.1. □

Let $\mathcal{A}$ denote the complement of $\mathcal{B}$, that is, $\mathcal{A}$ is the set of points (α_1, α_2) on the unit circle $\alpha_1^2 + \alpha_2^2 = 1$ which are *not* ε-bad (see the Key Definition). We want to give an upper bound to the integral

$$\int_{\mathcal{A}}\left(\int_{I^2}(A(T;\mathbf{y},(\alpha_1,\alpha_2)) - T\cdot\text{area}(A))^2\,d\mathbf{y}\right)ds, \tag{6.19}$$

where in the outer integral of (6.19) "ds" indicates integration with respect to the arc-length (since $\mathcal{A}$ is a "large" subset of the unit circle). We prove the following result.

Lemma 6.2 *We have*

$$\int_{\mathcal{A}}\left(\int_{I^2}(A(T;\mathbf{y},(\alpha_1,\alpha_2)) - T\cdot\text{area}(A))^2\,d\mathbf{y}\right)ds \le \text{area}(A)(1-\text{area}(A))\cdot\frac{2688}{\pi^2}\cdot\frac{1}{\varepsilon}\log_2 T\cdot(\log_2\log_2 T)^2.$$

Proof By using (6.6)–(6.7), we have

$$\text{integral}(6.19) \le \sum_{\substack{\mathbf{r}\in\mathbb{Z}^2:\\ \mathbf{r}\ne\mathbf{0}}}|a_{\mathbf{r}}|^2\cdot\int_{\mathcal{A}}\min\left\{\frac{1}{\pi^2(\alpha_1 r_1+\alpha_2 r_2)^2}, T^2\right\}ds. \tag{6.20}$$

If $(\alpha_1, \alpha_2) \in \mathcal{A}$ then by definition (see (6.8)–(6.9))

$$|\alpha_1 r_1 + \alpha_2 r_2| > \frac{\varepsilon}{40|\mathbf{r}|\cdot\log_2|\mathbf{r}|\cdot(\log_2\log_2|\mathbf{r}|)^2} \tag{6.21}$$

for all $|\mathbf{r}| \geq 8$ and

$$|\alpha_1 r_1 + \alpha_2 r_2| > \frac{\varepsilon}{40|\mathbf{r}|} \tag{6.22}$$

for all $1 \leq |\mathbf{r}| < 8$. Let $\mathbf{r} \in \mathbb{Z}^2$ with $|\mathbf{r}| = \sqrt{r_1^2 + r_2^2} \geq 8$ be arbitrary but fixed; to estimate the integral at the end of (6.20), we apply a standard power-of-two decomposition of the set

$$\mathcal{A}(\mathbf{r}) = \{(\alpha_1, \alpha_2) : \alpha_1^2 + \alpha_2^2 = 1,\ (6.21) \text{ holds}\} \supset \mathcal{A} \tag{6.23}$$

as follows: let ℓ be an arbitrary integer in the range

$$0 \leq \ell \leq L(\mathbf{r}) = \log_2 \left(\frac{40}{\varepsilon} |\mathbf{r}| \cdot \log_2 |\mathbf{r}| \cdot (\log_2 \log_2 |\mathbf{r}|)^2 \right), \tag{6.24}$$

and write

$$\mathcal{A}_\ell(\mathbf{r}) = \left\{(\alpha_1, \alpha_2) : \alpha_1^2 + \alpha_2^2 = 1,\ 2^{-\ell-1} < |\alpha_1 r_1 + \alpha_2 r_2| \leq 2^{-\ell}\right\}. \tag{6.25}$$

Finally, write

$$\mathcal{A}_{-1}(\mathbf{r}) = \left\{(\alpha_1, \alpha_2) : \alpha_1^2 + \alpha_2^2 = 1,\ |\alpha_1 r_1 + \alpha_2 r_2| > 1\right\}, \tag{6.26}$$

and so we have the disjoint decomposition

$$\mathcal{A}(\mathbf{r}) = \bigcup_{-1 \leq \ell \leq L(\mathbf{r})} \mathcal{A}_\ell(\mathbf{r}) \supset \mathcal{A}. \tag{6.27}$$

For every $\ell \geq 0$ we have the estimation

$$\text{measure}(\mathcal{A}_\ell(\mathbf{r})) \leq 4 \arcsin\left(\frac{1}{|\mathbf{r}| 2^\ell}\right) \leq 4\left(\frac{1}{|\mathbf{r}| \cdot 2^\ell} + \frac{1}{|\mathbf{r}|^2 \cdot 4^\ell}\right), \tag{6.28}$$

where (6.28) is just an easy adaptation of the argument at the beginning of the proof of Lemma 6.1.

Motivated by (6.20) and (6.27), we need to estimate the sum

$$\sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ |\mathbf{r}| \geq 8}} |a_\mathbf{r}|^2 \cdot \int_{\mathcal{A}(\mathbf{r})} \min\left\{\frac{1}{\pi^2(\alpha_1 r_1 + \alpha_2 r_2)^2}, T^2\right\} ds$$

$$= \sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ |\mathbf{r}| \geq 8}} |a_\mathbf{r}|^2 \sum_{\ell=-1}^{L(\mathbf{r})} \int_{\mathcal{A}_\ell(\mathbf{r})} \min\left\{\frac{1}{\pi^2(\alpha_1 r_1 + \alpha_2 r_2)^2}, T^2\right\} ds$$

$$\leq \sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ |\mathbf{r}| \geq 8}} |a_\mathbf{r}|^2 \sum_{\ell=-1}^{L(\mathbf{r})} \text{measure}(\mathcal{A}_\ell(\mathbf{r})) \cdot \min\left\{\frac{4^{\ell+1}}{\pi^2}, T^2\right\}$$

$$\leq \sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ |\mathbf{r}| \geq 8}} |a_\mathbf{r}|^2 \left(\sum_{\ell=0}^{L(\mathbf{r})} 4\left(\frac{1}{|\mathbf{r}| \cdot 2^\ell} + \frac{1}{|\mathbf{r}|^2 \cdot 4^\ell}\right) \min\left\{\frac{4^{\ell+1}}{\pi^2}, T^2\right\} + 2\pi \cdot \frac{1}{\pi^2}\right), \tag{6.29}$$

where in the last step we used (6.28); the last term in (6.29) is a trivial bound for the special case $\ell = -1$ in the summation; and finally, $L(\mathbf{r})$ is defined in (6.24).

To estimate (6.29), we need some rather long but totally routine calculations. For any $\mathbf{r} \in \mathbb{Z}^2$ with $|\mathbf{r}| \geq 8$, we have (δ is a 0-1 valued indicator function to be defined below)

$$\sum_{\ell=0}^{L(\mathbf{r})} \left(\frac{1}{|\mathbf{r}| \cdot 2^\ell} + \frac{1}{|\mathbf{r}|^2 \cdot 4^\ell}\right) \min\left\{\frac{4^{\ell+1}}{\pi^2}, T^2\right\}$$

$$= \frac{1}{|\mathbf{r}|} \sum_{\ell=0}^{L(\mathbf{r})} \min\left\{\frac{2^{\ell+2}}{\pi^2}, \frac{T^2}{2^\ell}\right\} + \frac{1}{|\mathbf{r}|^2} \sum_{\ell=0}^{L(\mathbf{r})} \min\left\{\frac{4}{\pi^2}, \frac{T^2}{4^\ell}\right\}$$

$$\leq \frac{1}{|\mathbf{r}|} \sum_{\substack{0 \leq \ell \leq L(\mathbf{r}): \\ 2^{\ell+1} \leq \pi T}} \frac{2^{\ell+2}}{\pi^2} + \frac{T^2}{|\mathbf{r}|} \sum_{\substack{0 \leq \ell \leq L(\mathbf{r}): \\ 2^{\ell+1} > \pi T}} 2^{-\ell} + \frac{\log_2 T}{|\mathbf{r}|^2}$$

$$\leq \frac{1}{|\mathbf{r}|} \cdot \frac{8}{\pi^2} \min\left\{2^{L(\mathbf{r})}, \pi T/2\right\} + \frac{T^2}{|\mathbf{r}|} \cdot 2\delta\left(2^{L(\mathbf{r})} \geq \pi T/2\right) \cdot \frac{2}{\pi T} + \frac{\log_2 T}{|\mathbf{r}|^2}, \tag{6.30}$$

where $\delta(2^{L(\mathbf{r})} \geq \pi T/2) = 1$ if $2^{L(\mathbf{r})} \geq \pi T/2$ and $\delta(2^{L(\mathbf{r})} \geq \pi T/2) = 0$ if $2^{L(\mathbf{r})} < \pi T/2$.

By (6.24), if $2^{L(\mathbf{r})} < \pi T/2$ then

$$\frac{1}{|\mathbf{r}|} \cdot \frac{8}{\pi^2} \min\left\{2^{L(\mathbf{r})}, \pi T/2\right\} + \frac{T^2}{|\mathbf{r}|} \cdot 2\delta\left(2^{L(\mathbf{r})} \geq \pi T/2\right) \cdot \frac{2}{\pi T}$$

$$= \frac{8}{\pi^2} \cdot \frac{40}{\varepsilon} \cdot \log_2 |\mathbf{r}| \cdot (\log_2 \log_2 |\mathbf{r}|)^2, \tag{6.31}$$

and if $2^{L(\mathbf{r})} \geq \pi T/2$ then

$$\frac{1}{|\mathbf{r}|} \cdot \frac{8}{\pi^2} \min\left\{2^{L(\mathbf{r})}, \pi T/2\right\} + \frac{T^2}{|\mathbf{r}|} \cdot 2\delta\left(2^{L(\mathbf{r})} \geq \pi T/2\right) \cdot \frac{2}{\pi T}$$

$$= \frac{4T}{\pi |\mathbf{r}|} + \frac{4T}{\pi |\mathbf{r}|} = \frac{8T}{\pi |\mathbf{r}|}. \tag{6.32}$$

If $2^{L(\mathbf{r})} < \pi T/2$ and $|\mathbf{r}| \geq 8$ then of course

$$\log_2 |\mathbf{r}| < L(\mathbf{r}) < \log_2(\pi T),$$

and so the last term in (6.31) can be estimated from above as follows:

$$\frac{8}{\pi^2} \cdot \frac{40}{\varepsilon} \cdot \log_2 |\mathbf{r}| \cdot (\log_2 \log_2 |\mathbf{r}|)^2 < \frac{8}{\pi^2} \cdot \frac{40}{\varepsilon} \log_2(\pi T) \cdot (\log_2 \log_2(\pi T))^2. \tag{6.33}$$

On the other hand, if we have the equality

$$\pi T/2 = 2^{L(\mathbf{r})} = \frac{40}{\varepsilon} |\mathbf{r}| \cdot \log_2 |\mathbf{r}| \cdot (\log_2 \log_2 |\mathbf{r}|)^2 \quad \text{and} \quad |\mathbf{r}| \geq 8, \tag{6.34}$$

then clearly

$$\frac{T}{|\mathbf{r}|} = \frac{2}{\pi} \cdot \frac{40}{\varepsilon} \cdot \log_2 |\mathbf{r}| \cdot (\log_2 \log_2 |\mathbf{r}|)^2 \leq \frac{2}{\pi} \cdot \frac{40}{\varepsilon} \cdot \log_2 T \cdot (\log_2 \log_2 T)^2, \tag{6.35}$$

and (6.35) remains true if we go beyond the equality (6.34) to the whole range $2^{L(\mathbf{r})} \geq \pi T/2$.

Summarizing, by (6.30)–(6.35) for any $\mathbf{r} \in \mathbb{Z}^2$ with $|\mathbf{r}| \geq 8$ we have

$$\sum_{\ell=0}^{L(\mathbf{r})} \left(\frac{1}{|\mathbf{r}| \cdot 2^\ell} + \frac{1}{|\mathbf{r}|^2 \cdot 4^\ell} \right) \min\left\{ \frac{4^{\ell+1}}{\pi^2}, T^2 \right\}$$
$$\leq \frac{16}{\pi^2} \cdot \frac{41}{\varepsilon} \log_2 T \cdot (\log_2 \log_2 T)^2. \tag{6.36}$$

Applying (6.36) in (6.29), we obtain

$$\sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ |\mathbf{r}| \geq 8}} |a_{\mathbf{r}}|^2 \cdot \int_{\mathcal{A}(\mathbf{r})} \min\left\{ \frac{1}{\pi^2(\alpha_1 r_1 + \alpha_2 r_2)^2}, T^2 \right\} ds$$
$$\leq \sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ |\mathbf{r}| \geq 8}} |a_{\mathbf{r}}|^2 \cdot \frac{64}{\pi^2} \cdot \frac{42}{\varepsilon} \log_2 T \cdot (\log_2 \log_2 T)^2. \tag{6.37}$$

Similarly,

$$\sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ 1 \leq |\mathbf{r}| < 8}} |a_{\mathbf{r}}|^2 \cdot \int_{\mathcal{A}(\mathbf{r})} \min\left\{ \frac{1}{\pi^2(\alpha_1 r_1 + \alpha_2 r_2)^2}, T^2 \right\} ds$$
$$\leq \sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ |\mathbf{r}| \geq 8}} |a_{\mathbf{r}}|^2 \cdot \frac{64}{\pi^2} \cdot \frac{42}{\varepsilon}. \tag{6.38}$$

Returning to (6.19)–(6.27), and using (6.37)–(6.38),

$$\frac{1}{2\pi} \int_{\mathcal{A}} \left(\int_{I^2} (A(T; \mathbf{y}, (\alpha_1, \alpha_2)) - T \cdot \text{area}(A))^2 \, d\mathbf{y} \right) ds$$
$$\leq \sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ |\mathbf{r}| \geq 8}} |a_{\mathbf{r}}|^2 \cdot \frac{1344}{\pi^3} \cdot \frac{1}{\varepsilon} \cdot \log_2 T \cdot (\log_2 \log_2 T)^2 + \sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ 1 \leq |\mathbf{r}| < 8}} |a_{\mathbf{r}}|^2 \cdot \frac{1344}{\pi^3} \cdot \frac{1}{\varepsilon}$$
$$\leq \sum_{\substack{\mathbf{r} \in \mathbb{Z}^2: \\ \mathbf{r} \neq \mathbf{0}}} |a_{\mathbf{r}}|^2 \cdot \frac{1344}{\pi^3} \cdot \frac{1}{\varepsilon} \log_2 T \cdot (\log_2 \log_2 T)^2$$
$$= \text{area}(A)(1 - \text{area}(A)) \cdot \frac{1344}{\pi^3} \cdot \frac{1}{\varepsilon} \log_2 T \cdot (\log_2 \log_2 T)^2, \tag{6.39}$$

where in the last step we used (6.2). Equation (6.39) gives Lemma 6.2. □

Now we are ready to finish the proof of Theorem 4: we just throw out the "bad" initial conditions and apply Chebyshev's inequality. First a definition: for any $\lambda > 0$ let

$$\Omega(\lambda) = \left\{ (\mathbf{y}, (\alpha_1, \alpha_2)) \in [0, 1)^2 \times \mathcal{A} : |A(T; \mathbf{y}, (\alpha_1, \alpha_2)) - T \cdot \text{area}(A)| \geq \lambda \right\}. \tag{6.40}$$

Combining (6.39)–(6.40) with Chebyshev's inequality,

$$\frac{1}{2\pi} \text{measure}(\Omega(\lambda)) \leq \text{area}(A)(1 - \text{area}(A)) \cdot \frac{1344}{\pi^3} \cdot \frac{1}{\varepsilon} \log_2 T \cdot (\log_2 \log_2 T)^2 \cdot \lambda^{-2}, \tag{6.41}$$

where "measure" stands for the 3-dimensional Lebesgue measure.

By making the choice

$$\lambda = \lambda_0 = \frac{10\sqrt{\text{area}(A)(1 - \text{area}(A))}}{\varepsilon} \sqrt{\log_2 T} \cdot \log_2 \log_2 T \tag{6.42}$$

in (6.41), we conclude

$$\frac{1}{2\pi} \text{measure}(\Omega(\lambda_0)) \leq \frac{\varepsilon}{2}. \tag{6.43}$$

If we throw out the set of initial conditions (starting point and angle) $(\mathbf{y}, (\alpha_1, \alpha_2))$ contained in $\Omega(\lambda_0)$, and also throw out those initial conditions $(\mathbf{y}, (\alpha_1, \alpha_2))$ for which the angle (α_1, α_2) is ε-bad (i.e., $(\alpha_1, \alpha_2) \in \mathcal{B}$), then by (6.43) and Lemma 6.1 the total loss is $\leq \varepsilon/2 + \varepsilon/2 = \varepsilon$. Combining this fact with (6.42)–(6.43), Theorem 4 follows.

Now we return to the long proof of Theorem 1.

7 Proof of Theorem 1: The Simplest Simultaneous Case

Let j, k be arbitrary integers with $1 \leq j < k \leq N$, and let

$$A_{j,k}(T) = A_{j,k}(T; \mathbf{y}_j, \mathbf{u}_j; \mathbf{y}_k, \mathbf{u}_k)$$

denote the total time between $0 < t < T$ when the jth torus-line $\mathbf{x}_j(t)$ and the kth torus-line $\mathbf{x}_k(t)$ are both in subset A simultaneously; in other words, when the two torus lines are in A *at the same time.*

The key observation is that we can describe $A_{j,k}(T)$ in terms of the Cartesian product $A \times A \subset I^6 = [0, 1]^6$ of $A \subset I^3$ with itself. Indeed, we have

$$A_{j,k}(T) = A_{j,k}(T; \mathbf{y}_j, \mathbf{u}_j; \mathbf{y}_k, \mathbf{u}_k)$$
$$= \text{measure}\left\{ t \in [0, T] : \mathbf{x}_j(t) \in A \text{ (mod 1) and } \mathbf{x}_k(t) \in A \text{ (mod 1)} \right\}$$
$$= \int_0^T \chi_A(\mathbf{x}_j(t)) \chi_A(\mathbf{x}_k(t)) \, dt = \int_0^T \chi_{A \times A}(\mathbf{x}_j(t), \mathbf{x}_k(t)) \, dt, \tag{7.1}$$

where $\chi_{A \times A}$ is the 0-1 valued characteristic function of $A \times A \subset I^6$. Write $B = A \times A$; we need the Fourier series of the characteristic function $\chi_B = \chi_{A \times A}$:

$$\chi_B(\mathbf{w}) = \chi_{A \times A}(\mathbf{w}) = \sum_{\mathbf{r} \in \mathbb{Z}^6} b_{\mathbf{r}} e^{2\pi i \mathbf{r} \cdot \mathbf{w}} \quad \text{with } b_{\mathbf{r}} = \int_{A \times A} e^{-2\pi i \mathbf{r} \cdot \mathbf{z}} \, d\mathbf{z}, \tag{7.2}$$

where $\mathbf{r} \cdot \mathbf{w} = r_1 w_1 + \cdots + r_6 w_6$ denotes the standard inner product. Clearly $b_0 = \text{vol}(A \times A) = \text{vol}^2(A)$ (= the volume of $A \times A$), and by Parseval's formula,

$$\sum_{\substack{\mathbf{r} \in \mathbb{Z}^6: \\ \mathbf{r} \neq \mathbf{0}}} |b_{\mathbf{r}}|^2 = \text{vol}^2(A) - \text{vol}^4(A), \tag{7.3}$$