Growth Rates of Hereditary Permutation Classes

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Counting functions — general setting

U is a countably infinite universum of some combinatorial structures with a partial ordering $(U, \prec)$ and a size function $s : U \to \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Hereditary class X
is a subset $X \subset U$ such that if $A \in U$, $B \in X$ and $A \prec B$, then $A \in X$.

Counting function of X
is the mapping $f : \mathbb{N}_0 \to \mathbb{N}_0$ where $f(n) = \#\{A \in X : \; s(A) = n\}$.

Which functions can be counting functions?

Example
$U = \{G = (\{1, 2, \ldots, n\}, E) : \; n \in \mathbb{N}_0\}$, $\prec$ ... subgraph relation, $s(G) = \#$ of vertices. $X = \{G : \; 2K_{1,1} \not\prec G\}$. Then $f(n) = \sum_{k=2}^{n} k\binom{n}{k} + 1 - \binom{n}{2} + \binom{n}{3}$.

Hereditary permutation classes

Now $U = \cup_{n=0}^{\infty} S_n$ where S_n is the set of $n!$ permutations of $\{1, 2, \ldots, n\}$. Size function: For $\pi \in S_n$ set $s(\pi) = n$.

Containment relation $\prec$
$\pi = a_1 a_2 \ldots a_n \prec \rho = b_1 b_2 \ldots b_m$: ρ has a subsequence $b_{i_1} b_{i_2} \ldots b_{i_n}$ such that $b_{i_j} < b_{i_k} \iff a_j < a_k$ for every $1 \le j, k \le n$.

Example
Relation $132 \prec \pi = a_1 a_2 \ldots a_n$ means that $a_i < a_k < a_j$ for some $1 \le i < j < k \le n$. $X = \{\pi \in U : 132 \not\prec \pi\}$ is of course hereditary and $f(n) = \frac{1}{n+1}\binom{2n}{n} \sim 4^n$.

Stanley–Wilf conjecture (1997)

If X is a hereditary class such that $X \neq U$, then $f(n) < c^n$ for a constant $c > 1$.

(For $X = U$ we have $f(n) = n!$.)

Why hereditary permutation classes?

- Interesting playground by itself. Some research directions: find $f(n)$ for particular hereditary classes, prove or disprove the Stanley-Wilf conjecture, determine which functions can be counting functions, generalize these problems to graphs and hypergraphs, ...

- Motivation from computer science.

  stack sortable permutations

  Knuth (TAOCP I, 1973): A permutation $\pi \in S_n$ is stack sortable if, when it is inputed to a stack and subjected to an appropriate sequence of 'push' and 'pop' operations, the stack can output it as $1, 2, \ldots, n$. Characterization:

  $$\pi \in S_n \text{ is stack sortable iff } \pi \not\succ 231.$$

  There are $\frac{1}{n+1}\binom{2n}{n}$ such permutations. Further results on sorting permutations by various devices: Pratt (1973), Tarjan (1973), ... , Zeilberger (1992): there are $\frac{2}{(3n+2)(3n+1)}\binom{3n+2}{2n+1}$ 2-stack-sortable permutations in S_n, ...

Why hereditary permutation classes?

- Motivation from algebraic geometry.

vexillary or 2143 avoiding permutations

Introduced by Lascoux $\&$ Schützenberger (1982). The Schubert polynomial indexed by a vexillary permutation is a flag Schur function.

321-hexagon avoiding permutations

Billey $\&$ Warrington (2000): 321-hexagon avoiding permutations in the theory of Schubert varieties and Kazhdan–Lustig polynomials. A permutation π is 321-hexagon avoiding if

$$\pi \not\succ \{321, 46718235, 46781235, 56718234, 56781234\}.$$

Stankova-Frenkel $\&$ West (2001): if $a_n = \#($321-hexagon avoiding $\pi \in S_n)$ then

$$a_n - 6a_{n-1} + 11a_{n-2} - 9a_{n-3} + 4a_{n-4} + 4a_{n-5} - a_{n-6} = 0$$

for $n \geq 6$ and thus $a_n \sim (3.43526\ldots)^n$.

So which functions can be counting functions of hereditary permutation classes?

For example, neither of the functions $f(n) = 70 + \lfloor 29 \cdot \sin(3n-2) \rfloor$, $f(n) = \lceil n^{5/8} \rceil$, and $f(n) = n - 11$ (for $n \geq 12$) is the counting function of a hereditary class, not even for $n > n_0$.

Theorem
Let $f(n)$ be the counting function of a hereditary class. If $f(n)$ is bounded then it is eventually constant. If $f(n)$ is unbounded then $f(n) \geq n$ for every n.

(Permutations $1, 2, \ldots, i-1, n, i, i+1, \ldots, n-1$ where $i, n \in \mathbb{N}_0$ form a hereditary class with $f(n) = n$.)

Theorem
Let $f(n)$ be the counting function of a hereditary class such that $f(n) < n^c$. Then for all $n > n_0$, $f(n) = \sum_{0 \leq i,j \leq m} a_{i,j} \binom{n-i}{j}$ where $m \in \mathbb{N}$ and $a_{i,j} \in \mathbb{Z}$ are constants.

What about exponential functions?

$(F_n)_{n\geq 1} = (1, 2, 3, 5, 8, 13, 21, 34, \ldots)$ is the sequence of Fibonacci numbers. $F_n \sim (1.61803\ldots)^n$.

Theorem
Let $f(n)$ be the counting function of a hereditary class. Then, for every $n \in \mathbb{N}$, $f(n) < n^c$ for a constant $c > 0$ or $f(n) \geq F_n$.

(The hereditary permutation class generated by the permutations $2, 1, 4, 3, 6, 5, \ldots, 2n, 2n-1$ has the counting function $f(n) = F_n$.)

Generalized Fibonacci numbers
$F_{n,k} = \#\{n = m_1 + m_2 + \cdots + m_r : \ 1 \leq m_i \leq k\}$. Recurrence: $F_{n,k} = 0$ for $n < 0$, $F_{0,k} = 1$, and $F_{n,k} = F_{n-1,k} + F_{n-2,k} + \cdots + F_{n-k,k}$ for $n > 0$. $F_{n,1} = 1$ and $F_{n,2} = F_n$.

Asymptotically
$F_{n,3} \sim (1.83928\ldots)^n$, $F_{n,4} \sim (1.92756\ldots)^n$, $F_{n,5} \sim (1.96594\ldots)^n$, $\ldots$, $F_{n,10} \sim (1.99901\ldots)^n$, $\ldots$ and $F_{n,k} \sim \alpha_k^n$ where $\alpha_k \to 2^-$.

Growth rates below 2^{n-1}

Theorem
Let $f(n)$ be the counting function of an infinite hereditary class. Then

- either $f(n) \geq 2^{n-1}$ for all $n \in \mathbb{N}$
- or there is a $k \in \mathbb{N}$ and a $c > 0$ such that $F_{n,k} \leq f(n) \leq n^c F_{n,k}$ for all $n \in \mathbb{N}$.

(The hereditary permutation class generated by the permutations $k, k-1, \dots, 1, 2k, 2k-1, \dots, k+1, \dots, kn, kn-1, \dots, kn-k+1$ has the counting function $f(n) = F_{n,k}$. The permutations

$$k_1, k_1 - 1, \dots, 1, k_1 + k_2, k_1 + k_2 - 1, \dots, k_1 + 1, \dots,$$
$$k_1 + \dots + k_r, k_1 + \dots + k_r - 1, \dots k_1 + \dots + k_{r-1} + 1$$

form a hereditary class with $f(n) = 2^{n-1}$.)

What are the growth rates above 2^{n-1}?