The Fundamental Theorem of Algebra

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Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{C}$ be the complex numbers. We present a proof in two steps of the *Fundamental Theorem of Algebra (FTA)* that every polynomial $p(z) \in \mathbb{C}[z]$ with $\deg p \ge 1$ has a root, that is, $p(\alpha) = 0$ for some $\alpha \in \mathbb{C}$. First we show that this follows easily if one knows that every complex number has a k-th root for every $k \in \mathbb{N}$. Then we prove topologically that $z \mapsto z^k$ maps the unit circle onto itself, which establishes existence of k-th roots. Finally, we give some comments and references.

Proposition 1. *If every binomial $z^k - a$, $k \in \mathbb{N}$ and $a \in \mathbb{C}$, has a root then so has every non-constant polynomial $p(z) \in \mathbb{C}[z]$.*

Proof. Let $p(z) \in \mathbb{C}[z]$ with $\deg p \ge 1$ be given. First we prove that $|p(z)|$ attains on $\mathbb{C}$ minimum value. We write

$$\begin{aligned} p(z) &= a_n z^n + a_{n-1} z^{n-1} + \dots + a_0 = z^n(a_n + a_{n-1}z^{-1} + \dots + a_0 z^{-n}) \\ &= z^n(a_n + q(z)) \end{aligned}$$

where $n \ge 1$, $a_n \ne 0$, and $\lim_{|z|\to\infty} q(z) = 0$. Let $R > 0$ be so large that $R^n|a_n|/2 \ge |p(0)|$ and $|z| > R \Rightarrow |q(z)| \le |a_n|/2$, and let $D = \{z \in \mathbb{C} \mid |z| \le R\}$. The disc D is compact and $|p(z)|$ is a continuous mapping from $\mathbb{C}$ to $[0, +\infty)$, hence an $\alpha \in D$ exists such that $|p(\alpha)| \le |p(z)|$ for every $z \in D$. But for $z \in \mathbb{C}\backslash D$ this holds too because $0 \in D$:

$$|p(\alpha)| \le |p(0)| \le R^n|a_n|/2 < |z|^n(|a_n| - |q(z)|) \le |p(z)| \ .$$

So $|p(\alpha)| \le |p(z)|$ for every $z \in \mathbb{C}$.

Next we show that $|p(\alpha)| = 0$ and so α is a root of $p(z)$. For contradiction, let $|p(\alpha)| > 0$. We reexpand $p(z)$ in the basis $(z-\alpha)^j$, $j = 0, 1, \dots$:

$$\begin{aligned} p(z) &= b_0 + b_k(z-\alpha)^k + b_{k+1}(z-\alpha)^{k+1} + \dots + b_n(z-\alpha)^n \\ &= b_0 + b_k(z-\alpha)^k + r(z) \end{aligned}$$

where $b_0, b_k, b_n \ne 0$, $1 \le k \le n$, and $\lim_{z\to\alpha} r(z)/(z-\alpha)^k = 0$. In fact, $b_0 = p(\alpha) \ne 0$ and $b_n = a_n$. We set, invoking the hypothesis,

$$\beta = \alpha + \delta(-b_0/b_k)^{1/k}$$

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where $(-b_0/b_k)^{1/k}$ is a k-th root of $-b_0/b_k$ and $\delta > 0$ is real and small enough so that $\delta^k \le 1$ and $|r(\beta)| \le \delta^k |b_0|/2$. So $\beta \ne \alpha$ and

$$\begin{aligned} |p(\beta)| &= |b_0 - \delta^k b_0 + r(\beta)| \le |b_0 - \delta^k b_0| + |r(\beta)| \\ &= |b_0| - \delta^k |b_0| + |r(\beta)| \le |b_0| - \delta^k |b_0|/2 \\ &< |b_0| = |p(\alpha)| \ , \end{aligned}$$

in contradiction with the minimality of $|p(\alpha)|$. □

We see that the FTA follows if we show that every $a \in \mathbb{C}$ has a k-th root for every $k \in \mathbb{N}$. For $a = 0$ or real $a > 0$ it is clear. Thus by replacing $a \ne 0$ with $a/|a|$ we may restrict to $a \in S$, the unit complex circle $S = \{z \in \mathbb{C} \mid |z| = 1\}$. For any $k \in \mathbb{N}$, the mapping

$$\mathbb{C} \ni z \mapsto z^k \in \mathbb{C}$$

maps S to itself. It is clearly continuous on $\mathbb{C}$ and locally injective on $\mathbb{C}\backslash\{0\}$, in the sense that for every nonzero $z \in \mathbb{C}$ its restriction to a neighborhood of z is injective. This follows easily from the factorization

$$u^k - v^k = (u - v)(u^{k-1} + u^{k-2}v + \cdots + v^{k-1})$$

— if $u, v \in \mathbb{C}$ are distinct and near enough to a nonzero $z \in \mathbb{C}$, the second factor is almost $kz^{k-1} \ne 0$, and so $u^k \ne v^k$. The next proposition thus shows that every complex number has a k-th root and completes the proof of the FTA.

Proposition 2. *If $f : S \to S$ is a continuous and locally injective mapping of the unit circle to itself then $f(S) = S$.*

Proof. Suppose that $f : S \to S$ is continuous. We show that $f(S) = S$ or f is not locally injective. Since S is compact and connected, $f(S)$ is a nonempty, closed and connected subset of S. It follows that $f(S)$ is (i) a single point of S or (ii) a closed arc of S with two distinct endpoints or (iii) the whole S. The cases (i) and (iii) are clear. Suppose that the case (ii) occurs and the endpoints of the arc $f(S)$ are b and c. We take an $a \in S$ with $f(a) = b$. Let $A \subset S$ be any closed arc containing a in its interior. Then $f(A) = B \subset S$ is a closed arc such that $f(a) = b$ is one of its two endpoints. If the restriction of f to A is injective, it is a homeomorphism of A and B (since A is compact). But this is impossible because $A\backslash\{a\}$ is disconnected but $B\backslash\{f(a)\}$ is connected. Thus $f \,|\, A$ is not injective and we deduce that f is not locally injective near a. □

This text arose from my teaching of "Matematická analýza III" in fall of 2015. I have found it useful and pleasing to record a simple and selfcontained proof of the FTA, and used it in my lectures to illustrate topological notions such as continuity, compactness, closeness, homeomorphism, interior, and (dis)connectivity. Proofs of the FTA are an often, maybe too often, trodden area — search in the *Mathematical Reviews* database returns (in October

2015) 128 items whose title contains “the fundamental theorem of algebra”, of which I randomly selected and printed and tried to look at Baltus [2], Derksen [3], Lazer and Leckband [4], Moritz [5], Pascu [6], Sheffer [9], Shipman [10], and Suzuki [11] — and so I put here the usual disclaimer of claiming not much originality in the above proof. I think I remember well that I learned the minimization argument proving Proposition 1 from Zorich [12, Chapter 5.5.5]. This minimization proof of the FTA goes back to Argand [1]. I also think I remember well not learning Proposition 2 from anywhere. I want to remark that this minimization proof is in fact not so much a proof of the FTA as a reduction from general non-constant polynomials $p(z)$ to binomial polynomials $z^k - a$, as I stressed by the formulation of Proposition 1. Usually, say in Zorich [12, Chapter 5.5.5] or Rudin [7, Chapter 8, p. 184], this is not emphasized and is buried in and obscured by notation. But the truth is that the second necessary step, proving existence of k-th roots, is equally important and at least as difficult as the previous minimization argument. One can obtain them by de Moivre’s formula via the sinus–cosinus machinery, which means to establish properties of the exponential function $z \mapsto \sum_{n\geq 0} z^n/n!$ (especially that its image is $\mathbb{C}\backslash\{0\}$) which is very nicely done in Rudin [8, Prologue]. One can probably calculate k-th roots by converging power series solutions of some equations, but then one can try to find in this way solution to the general equation $p(z) = 0$, and already Sheffer [9] thought about that (and much earlier before him d’Alembert, see Baltus [2]). Or one can use topological arguments, as we did in Proposition 2. Certainly one can write a formula for the 2-nd root of any $a \in \mathbb{C}$. Is there a more algebraic argument proving existence of k-th roots?

References

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