There is no free will: almost every value of every function is predetermined by earlier values

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For $n \in \mathbb{N} = \{1, 2, \dots\}$, $f : [n] = \{1, 2, \dots, n\} \to \mathbb{R}$ and $a \in [n]$, we denote by $f \,|\, a$ the restriction of f to $[a-1]$ (we set $f \,|\, 1 = \emptyset$). Can one determine the value $f(a)$ from $f \,|\, a$? More formally, for given $n \in \mathbb{N}$ a *predictor* P is a mapping

$$P : \{f : [a-1] \to \mathbb{R} \mid a \in [n]\} \to \mathbb{R}$$

from the set of all real functions whose domains are proper initial segments of $[n]$ to $\mathbb{R}$, and we say that for a given function $f : [n] \to \mathbb{R}$ and $a \in [n]$ a predictor P *predicts the value* $f(a)$ if $P(f \,|\, a) = f(a)$, and otherwise that P *errs at* $f(a)$. Is there a predictor that for every function f predicts at least some of its values? Of course not, as everybody sees.

Proposition 1. *Let $n \in \mathbb{N}$. Then for every predictor P there is a function $f : [n] \to \mathbb{R}$ such that P errs at $f(a)$ for every $a \in [n]$.*

Proof. For P and $a = 1, 2, \dots, n$, we define f by induction as $f(a) := P(f \,|\, a) + 1$, say. Then $P(f \,|\, a) \neq f(a)$ for every $a \in [n]$. □

We replace the discrete domain $[n]$ with $\mathbb{R}$ and consider functions $f : \mathbb{R} \to \mathbb{R}$ and for $a \in \mathbb{R}$ their restrictions $f \,|\, a$ to $(-\infty, a)$. A *predictor* P is a mapping

$$P : \{f : (-\infty, a) \to \mathbb{R} \mid a \in \mathbb{R}\} \to \mathbb{R}$$

from the set of all real functions defined before an $a \in \mathbb{R}$ to $\mathbb{R}$. Again, for $f : \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$, a predictor P *predicts the value* $f(a)$ if $P(f \,|\, a) = f(a)$ and it *errs at* $f(a)$ if $P(f \,|\, a) \neq f(a)$. Unlike Proposition 1, now we have the following at first surprising and counterintuitive result that a predictor exists that for every function $f : \mathbb{R} \to \mathbb{R}$ predicts almost every value. (Note that for the class of continuous functions $f : \mathbb{R} \to \mathbb{R}$ the obvious predictor predicts for every function every value.)

Proposition 2. *If we assume the axiom of choice, there exists a predictor P that for every function $f : \mathbb{R} \to \mathbb{R}$ errs only at at most countably many values.*

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Proof. We denote by M the set of all functions $f: \mathbb{R} \to \mathbb{R}$. By the axiom of choice there is a well ordering $(M, \prec)$. For $a \in \mathbb{R}$ and $g: (-\infty, a) \to \mathbb{R}$ we define our miraculous predictor P by

$$P(g) := f_0(a) \quad \text{where} \quad f_0 = \min(\{f \in M \mid f \,|\, a = g\}) ,$$

the minimum taken with respect to $\prec$. We show that P has the stated property.

Let $f \in M$ be arbitrary and

$$X = \{a \in \mathbb{R} \mid P(f \,|\, a) \neq f(a)\}$$

be the arguments where P did not predict correctly the value $f(a)$. We claim that if $a, b \in X$ with $a < b$, $f_0 = \min(\{g \in M \mid g \,|\, a = f \,|\, a\})$, and $f_1 = \min(\{g \in M \mid g \,|\, b = f \,|\, b\})$, then $f_0 \prec f_1$. Indeed, the former set of gs contains the latter one as $(-\infty, a) \subset (-\infty, b)$, so $f_0 \preceq f_1$, but $f_0 \neq f_1$ because $f_0(a) = P(f \,|\, a) \neq f(a) = (f \,|\, b)(a) = f_1(a)$. Thus any infinite strictly descending chain $\cdots < a_2 < a_1$ in $(X, <)$ would give an infinite strictly descending chain in $(M, \prec)$, which cannot be since $(M, \prec)$ is a well ordering. Hence $(X, <)$ is a well ordered subset of $\mathbb{R}$, which implies that X is finite or countable (the mapping $h: X \to \mathbb{Q}$ defined by $h(\max X) =$ any $\alpha \in \mathbb{Q}$ larger than $\max X$ and, for $x \in X$ not the maximum of X, by $h(x) =$ any $\alpha \in \mathbb{Q}$ between x and $\min(\{y \in X \mid y > x\})$, is an injection). □

So, ironicly, there is no free will when we accept the axiom of choice: if we are building a function $f: \mathbb{R} \to \mathbb{R}$ step by step by going with $a \in \mathbb{R}$ from $-\infty$ to $+\infty$, then except for at most countably many flashes of free will, at the remaining instances $f(a)$ is completely predetermined by the earlier values $f(x)$, $x < a$, and we have no choice!

Proposition 2 is due to Hardin and Taylor [2, Teorem 3.1], for further results in this spirit see Bajpai and Velleman [1].

References

[1] D. Bajpai and D. J. Velleman, Anonymity in predicting the future, ArXiv:1508.06865v1, 12 pages, 2015.

[2] Ch. S. Hardin and A. D. Taylor, A peculiar connection between the axiom of choice and predicting the future, *Amer. Math. Monthly* **115** (2008), 91–96.