Birch's theorem: if $f(n)$ is multiplicative and has a non-decreasing normal order then $f(n) = n^\alpha$

Martin Klazar*

June 23, 2016

1 Introduction

In 1967 B. J. Birch, later of the Birch and Swinnerton-Dyer conjecture fame, proved in [2] a most interesting result.

Theorem (Birch, 1967). *The only multiplicative functions $f : \mathbb{N} \to \mathbb{R}_{\ge 0}$ that are unbounded and have a non-decreasing normal order are the powers of n, the functions $f(n) = n^\alpha$ for a constant $\alpha > 0$.*

Multiplicativity means that $f(mn) = f(m)f(n)$ for every two coprime numbers $m, n \in \mathbb{N}$ (thus $f(1) = 1$ unless $f \equiv 0$), $\mathbb{N} = \{1, 2, \dots\}$, and the clause about a non-decreasing normal order means that a non-decreasing function $g : \mathbb{N} \to \mathbb{R}_{>0}$ exists such that for every $\varepsilon > 0$, $\#(n \le x \mid \frac{f(n)}{g(n)} \notin (1-\varepsilon, 1+\varepsilon)) = o(x)$ as $x \to +\infty$.

In this write-up I present the proof of Birch's theorem, as given in Birch [2] and Narkiewicz [13, pp. 98–102] (see also [14]). It is a beautiful proof in the erdősian style. To be honest, I started with the intention to correct two errors I thought I had discovered in the argument. Fortunately, in the process of writing everything clarified and the errors disappeared. Still, I will point out the two steps I struggled with. To the interested reader, much smarter than me, they will certainly pose no difficulty.

2 The proof with two conundrums

We use notation of [2], so let

$$b(n) = \log f(n) \ \text{ and } \ c(n) = \log g(n) \ .$$

Birch [2, p. 149] writes just "If f is unbounded, then $g(n)$ tends to infinity with n, so we may suppose that $c(n) > 0$ for all n." but Narkiewicz [13,

*klazar@kam.mff.cuni.cz

Lemat 2.5 on p. 98] gives more details. Assume for contrary that $g(n)$ has a finite limit $a > 0$. Then, by the relation bounding f and g, there are constants $0 < A < a < B$ such that for every $x > 0$ and $n \le x$ we have $A < f(n) < B$, with $o(x)$ exceptions. Let $E \subset \mathbb{N}$ be the exceptions; E has density 0. Fix any $M > B$. Since f is unbounded, there is an $m \in \mathbb{N}$ with $f(m) > M/A$. The sets $\{nm+1 \mid n \in \mathbb{N}\}$ and $\{(nm+1)m \mid n \in \mathbb{N}\}$ have positive densities and thus so has $X = \{n \in \mathbb{N} \mid nm+1, (nm+1)m \notin E\}$. For any $n \in X$ we get the contradiction $B > f((nm+1)m) = f(nm+1)f(m) > Af(m) > M$.

Thus indeed $\lim g(n) = +\infty$. Changing finitely many values of $g(n)$ we may assume that always $g(n) > 1$ and $c(n) > 0$. By Birch [2], "Using the three conditions

given $\varepsilon > 0$, $|b(n) - c(n)| < \varepsilon$ for all but $o(x)$ integers $n < x$;
$b(mn) = b(m) + b(n)$ if $(m, n) = 1$;
$c(n) \ge c(m) > 0$ for $n \ge m$;

we gradually deduce more and more till everything collapses." Let $m, n \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary with $|b(m) - c(m)|, |b(n) - c(n)| < \varepsilon$. We assume that $m, n \ge 2$. It follows that for any $\eta \in (0, \frac{1}{2})$ there is an $S > 0$ such that for every $R \ge S$ there are $s, t \in \mathbb{N}$ satisfying

$$(1-\eta)R < s < R < t < (1+\eta)R,\ s \equiv t \equiv 1 \pmod{mn}$$

and

$$|b(s) - c(s)|,\ |b(ms) - c(ms)|,\ |b(t) - c(t)|,\ |b(nt) - c(nt)| < \varepsilon\ .$$

(Only $o(R)$ of the integers $s \in ((1-\eta)R, R)$ violate the first or the second lastly displayed inequality, and so for large R we certainly find there an $s \equiv 1 \pmod{mn}$ satisfying both. The same for t.) From $b(ms) = b(m) + b(s)$ and $b(nt) = b(n) + b(t)$ we get

$$|c(ms) - c(m) - c(s)|,\ |c(nt) - c(n) - c(t)| < 3\varepsilon\ .$$

We define by induction numbers $s_0 < s_1 < \dots$ and $t_0 < t_1 < \dots$ in $\mathbb{N}$, all congruent to 1 modulo mn, such that

$$(1-\eta)S < s_0 < S < t_0 < (1+\eta)S$$

and, for every $i, j \in \mathbb{N}_0$,

$$(1-\eta)ms_i < s_{i+1} < ms_i,\ nt_j < t_{j+1} < (1+\eta)nt_j\ ,$$

and

$$|b(s_i) - c(s_i)|,\ |b(ms_i) - c(ms_i)|,\ |b(t_j) - c(t_j)|,\ |b(nt_j) - c(nt_j)| < \varepsilon\ .$$

(In the previous claim we first set $R = S$ and get $s_0 = s$, then we set $R = ms_0 (\ge S)$ and get $s_1 = s$, and so on. Since $m \ge 2$ and $\eta < \frac{1}{2}$, we stay above S

and s_i increase. Similarly and more easily for t_j.) Then, as we know, for every $i \in \mathbb{N}_0$ one has

$$|c(ms_i) - c(m) - c(s_i)| < 3\varepsilon .$$

Monotonicity of c gives

$$c(s_i) > c(ms_i) - c(m) - 3\varepsilon \geq c(s_{i+1}) - c(m) - 3\varepsilon$$

and so $c(s_h) < c(S) + hc(m) + 3h\varepsilon$ for every $h \in \mathbb{N}$ by iteration. On the other hand, $s_h > (1-\eta)^{h+1} m^h S$ by iterating the above inequalities. Similarly for t_j we get $c(t_k) > c(S) + kc(n) - 3k\varepsilon$ for every $k \in \mathbb{N}$ and $t_k < (1+\eta)^{k+1} n^k S$.

Now if $h, k \in \mathbb{N}$ are such that $m^h > n^k$, equivalently $h \log m > k \log n$ (recall that $\log m \neq 0$), we may select $\eta > 0$ so small that still

$$(1-\eta)^{h+1} m^h > (1+\eta)^{k+1} n^k .$$

This implies that $s_h > t_k$ and $c(s_h) \geq c(t_k)$ (by monotonicity of c), hence $hc(m) + 3h\varepsilon > kc(n) - 3k\varepsilon$ and

$$\frac{h}{k} > \frac{c(n) - 3\varepsilon}{c(m) + 3\varepsilon} .$$

It follows that

$$\frac{\log n}{\log m} \geq \frac{c(n) - 3\varepsilon}{c(m) + 3\varepsilon} .$$

(But how come? *This is the first step I struggled with.* Don't we assume that $h/k > (\log n)/(\log m)$? To combine inequalities by transitivity we would need this one be opposite!)

Nevertheless, we get

$$\frac{c(n)}{\log n} - \frac{c(m)}{\log m} \leq 3\varepsilon \left(\frac{1}{\log m} + \frac{1}{\log n} \right)$$

and, changing the roles of m and n, the reverse inequality $\cdots \geq -3\varepsilon \ldots$. So we have proved that

$$\left| \frac{c(n)}{\log n} - \frac{c(m)}{\log m} \right| \leq 3\varepsilon \left(\frac{1}{\log m} + \frac{1}{\log n} \right)$$

whenever $|b(m) - c(m)| < \varepsilon$ and $|b(n) - c(n)| < \varepsilon$. This implies

$$\left| \frac{c(n)}{\log n} - \frac{c(m)}{\log m} \right| \leq (|b(m) - c(m)| + |b(n) - c(n)|) \left(\frac{3}{\log m} + \frac{3}{\log n} \right)$$

for all m, n. (But how come? *This is the second step I struggled with.* Let's say that the penultimate displayed inequality holds for every m, n as an equality for 3ε replaced with 2ε, and that we have m, n such that $|b(m) - c(m)|, |b(n) - c(n)| < \varepsilon/4$. The last two displayed inequalities then contradict each other!).

Nevertheless, we conclude the proof. Obviously, $|b(n_i) - c(n_i)| \to 0$ for a sequence $n_1 < n_2 < \dots$. The last displayed inequality shows that the values $c(n_i)/\log n_i$ are bounded. Passing to a subsequence we get $\lim_i c(n_i)/\log n_i = \alpha$, with a finite limit α. Setting $n = n_i$ and letting $i \to \infty$ gives

$$|c(m) - \alpha \log m| \le 3|b(m) - c(m)| \ \text{ and } \ |b(m) - \alpha \log m| \le 4|b(m) - c(m)|$$

for every $m \in \mathbb{N}$ (well, $m \ge 2$). Thus, given any $\varepsilon > 0$, $|b(m) - \alpha \log m| < \varepsilon$ for all but $o(x)$ numbers $m \le x$. Let $E \subset \mathbb{N}$ be the set of exceptional m; it has density 0. We take any $m \in \mathbb{N}$. The set $X = \{n \in \mathbb{N} \mid (n, m) = 1, n, mn \notin E\}$ has positive density. For any $n \in X$ we have

$$|b(n) - \alpha \log n|, \ |b(mn) - \alpha \log(mn)| < \varepsilon \ .$$

So, by the additivity of the functions b and log, $\varepsilon > |b(mn) - \alpha \log(mn)| \ge |b(m) - \alpha \log m| - |b(n) - \alpha \log n|$ and $|b(m) - \alpha \log m| < 2\varepsilon$. As this holds for any $\varepsilon > 0$, we get the desired equality

$$b(m) = \alpha \log m \ \text{ or } \ f(m) = m^\alpha$$

for every $m \in \mathbb{N}$. We are done. Well, ...

3 Concluding remarks

How do we resolve the two conundrums? In the first we have three real quantities $a = h/k$, $b = (\log n)/(\log m)$, and $c = (c(n) - 3\varepsilon)/(c(m) + 3\varepsilon)$ and we know that $a > b \Rightarrow a > c$. From $b > a, a > c$ we would get $b > c$ by transitivity. However, in our situation also $a > b \Rightarrow a > c$ implies $b \ge c$, via a more subtle argument relying on the density of $\mathbb{Q}$ in $\mathbb{R}$. The point is that we may select a larger than b and as close to b as we wish. Assume for contrary that $c > b$. Then we select a in-between as $c > a > b$, and $a > b \Rightarrow a > c$ gives $a > c$, a contradiction. Thus $b \ge c$. The second conundrum is more psychological and stems from assuming $\varepsilon > 0$ to be a fixed thing. But if we drop it and regard ε as a variable on par with m, n, everything is clear. We know that $|b(m) - c(m)|, |b(n) - c(n)| < \varepsilon \Rightarrow |\frac{c(n)}{\log n} - \frac{c(m)}{\log m}| \le 3\varepsilon(\frac{1}{\log m} + \frac{1}{\log n})$. Thus for $m, n \in \mathbb{N}$ (and $m, n \ge 2$) we just set $\varepsilon = |b(m) - c(m)| + |b(n) - c(n)|$ and the implication yields the stated conclusion (perturbing g a little bit we may assume that $|b(n) - c(n)| > 0$ for every $n \in \mathbb{N}$).

Birch's article [2] is cited in [1, 3, 4, 6, 7, 8, 9, 10, 11, 13, 14].

It all started when I read the recent preprint of Shiu [18] that reproves Segal's result [16, 17] that Euler's function $\varphi(n)$ does not have non-decreasing normal order, as a corollary of the next nice theorem.

Theorem (Shiu, 2016; Segal, 1964). *If $f : \mathbb{N} \to \mathbb{R}_{\ge 0}$ has a non-decreasing normal order, $f(n) = O(n)$, and $\sum_{n \le x} f(n) \sim Ax^2/2$ and $\sum_{n \le x} f(n)^2 \sim Bx^3/3$ as $x \to +\infty$ for some constants $A, B > 0$, then $A^2 \ge B$.*

For $f(n) = \varphi(n)$ (which is $O(n)$) we have $A = \prod_p (1 - p^{-2})$ and $B = \prod_p (1 - 2p^{-2} + p^{-3})$ (see [18] for proofs of these average orders). Since $A^2 < B$, we conclude that $\varphi(n)$ does not have non-decreasing normal order. It follows also from Birch's theorem, since $\varphi(n)$ is multiplicative (and unbounded). For results on sets where $\varphi(n)$ itself is monotonous see Pollack, Pomerance, and Treviño [15].

Finally, I was inspired by all this and the discussion at [19] to pose the following problem.

Problem (MK, 2016). *Does $\varphi(n)$ have an effective normal order? That is, is there a function $g : \mathbb{N} \to \mathbb{N}$ such that for every $\varepsilon > 0$, $\#(n \le x \mid \frac{\varphi(n)}{g(n)} \notin (1-\varepsilon, 1+\varepsilon)) = o(x)$ as $x \to +\infty$, and*

one can compute $n \mapsto g(n)$ in time polynomial in $\log n$?

References

[1] J.-P. Allouche, M. Mendès France, and J. Peyrière, Automatic Dirichlet series, J. Number Theory 81 (2000) 359–373.

[2] B. J. Birch, Multiplicative functions with non-decreasing normal order, J. London Math. Soc. 42 (1967) 149–151.

[3] P. D. T. A. Elliott, On a conjecture of Narkiewicz about functions with non-decreasing normal order, Colloq. Math. 36 (1976) 289–294.

[4] P. D. T. A. Elliott, *Probabilistic Number Theory. I. Mean-value Theorems*, Springer-Verlag, New York–Berlin, 1979.

[5] P. D. T. A. Elliott, *Arithmetic Functions and Integer Products*, Springer-Verlag, New York, 1985.

[6] P. Erdős and C. Ryavec, A characterization of finitely monotonic additive functions, J. London Math. Soc. 5 (1972), 362–367.

[7] J.-M. de Koninck, Review of [5] and [12], Bull. Amer. Math. Soc. 18 (1988), 230–247.

[8] J.-M. de Koninck, N. Doyon, and P. Letendre, On the proximity of additive and multiplicative functions, Funct. Approx. Comment. Math. 52 (2015) 327–344.

[9] J.-M. de Koninck and F. Luca, *Analytic Number Theory. Exploring the Anatomy of Integers*, American Mathematical Society, Providence, RI, 2012.

[10] K. Kovács, On the characterization of additive and multiplicative functions, Studia Sci. Math. Hungar. 18 (1982) 1–11.

[11] L. Matthiesen, Correlations of the divisor function, Proc. Lond. Math. Soc. 104 (2012) 827–858.

[12] P. J. McCarthy, *Introduction to Arithmetical Functions*, Springer-Verlag, New York, 1986.

[13] W. Narkiewicz, *Teoria liczb*, Państwowe Wydawnictwo Naukowe, Warszawa, 1990 (in Polish).

[14] W. Narkiewicz, *Number Theory*, World Scientific Publishing Co., Singapore, 1983 (translated from the 1977 edition of [13] by S. Kanemitsu).

[15] P. Pollack, C. Pomerance, and E. Treviño, Sets of monotonicity for Euler's totient function, Ramanujan J. 30 (2013) 379–398.

[16] S. L. Segal, A note on normal order and the Euler φ-function, J. London Math. Soc. 39 (1964) 400–404.

[17] S. L. Segal, On non-decreasing normal orders, J. London Math. Soc. 40 (1965) 459–466.

[18] P. Shiu, On functions without a normal order, preprint, arXiv:1606.04533, June 2016, 4 pages.

[19] How hard is it to compute the Euler totient function?, http://mathoverflow.net/questions/3274/

Charles University, KAM MFF UK, Malostranské nám. 25, 11800 Praha, Czechia