L8 (April 24, 2020) We prove: ①

Thm. (von Staudt–Clausen) $\forall$ even $k \in \mathbb{N}$:
$$B_k + \sum_{p,\,(p-1)\mid k} \frac{1}{p} \in \mathbb{Z}$$
(p is a prime number, $\mid$ denotes the divisibility relation and B_k are the Bernoulli numbers).

Let, for $k, n \in \mathbb{N}$, $S_k(n) := 1^k + 2^k + \dots + (n-1)^k$. As is well known, $S_1(n) = 1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2} = \binom{n}{2}$. The B. numbers appear in a generalization of this formula:

Proposition (*) $S_k(n) =$
$$\boxed{B} \quad = \sum_{r=0}^{k} \binom{k}{r} \frac{B_r}{k+1-r} n^{k+1-r},$$
which is a polynomial in n with degree $k+1$.

Proof. We have the formal power series identity
$$1 + e^x + e^{2x} + \dots + e^{(n-1)x} = \frac{e^{nx}-1}{x} \cdot \frac{x}{e^x-1}.$$
The coeff. of $\frac{x^k}{k!}$ on the left side is, when we re-

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(*) "the once well-known formula", in the words of J. Cassels.

Call that $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \dots$, exactly $S_k(n)$. ②

On the right side: $\underbrace{[\frac{x^k}{k!}] \frac{e^{nx}-1}{x} \cdot \frac{x}{e^x-1}} =$

$[x^m]F(x)$ denotes the coeff. of x^m in the expansion of $F(x)$.

$$= \left[\frac{x^k}{k!}\right] \sum_{j=1}^{\infty} \frac{n^j x^{j-1}}{j!} \cdot \sum_{l=0}^{\infty} \frac{B_l x^l}{l!} =$$

$= (\text{We set } j+l-1 = k) = \cancel{\left[\frac{x^k}{k!}\right]} \sum_{j=1}^{k+1} k! \frac{n^j}{j!} \frac{B_{k-j+1}}{(k-j+1)!} =$

$$= \cancel{\left[\frac{x^k}{k!}\right]} \sum_{j=1}^{k+1} \cancel{\frac{k!}{k!}} \cdot \frac{1}{k-j+1} \binom{k}{j} n^j B_{k-j+1} =$$

$$= \sum_{j=1}^{k+1} \frac{1}{k-j+1} \binom{k}{j} n^j B_{\underbrace{k-j+1}_{r}} = \sum_{r=0}^{k} \frac{1}{r} \binom{k}{r-1} \underbrace{n^{k+1-r}} B_r =$$

$$= \sum_{r=0}^{k} \frac{1}{k-r+1} \binom{k}{r} n^{k+1-r} B_r \quad \boxtimes$$

So, for example, $S_5(n) = 1^5 + 2^5 + \dots + (n-1)^5 = \sum_{r=0}^{5} \frac{1}{6-r} \binom{5}{r} B_r n^{6-r} =$

$$= \frac{1}{6} B_0 n^6 + \frac{1}{5}\binom{5}{1} B_1 n^5 + \frac{1}{4}\binom{5}{2} B_2 n^4 + \frac{1}{2}\binom{5}{4} B_4 n^2 = \frac{n^6}{6} - \frac{1}{2} n^5 + \frac{5}{12} n^4 - \frac{n^2}{12} =$$

$$\boxed{= \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2.}$$

For a prime number p and $n \in \mathbb{Z}$, $n \neq 0$, we set $\nu_p(n) :=$ the maximum $k \in \mathbb{N}_0$ s.t. $p^k \mid n$, and $\nu_p(0) := +\infty$. For a fraction $\alpha = \frac{a}{b} \in \mathbb{Q}$ we set $\nu_p(\alpha) = \nu_p(a) - \nu_p(b)$. Finally, for any real constant $c \in (0,1)$ we define the <u>p-adic norm</u> $|\alpha|_p$ of $\alpha \in \mathbb{Q}$ as

$$\boxed{|\alpha|_p := c^{\nu_p(\alpha)}.}$$

One usually sets $c = \frac{1}{p}$ but here it is not important.

Proposition $\forall \alpha, \beta \in \mathbb{Q}$ we have:

1. $|\alpha|_p \geq 0$ and $= 0 \iff \alpha = 0$
2. $|\alpha\beta|_p = |\alpha|_p \cdot |\beta|_p$
3. $\underbrace{|\alpha+\beta|_p \leq \max(|\alpha|_p, |\beta|_p)}_{\text{the strong triangle inequality}} \; (\leq |\alpha|_p + |\beta|_p)$

Proof – if you want, an exercise.

With $d(\alpha,\beta) := |\alpha - \beta|_p$ we get the metric space $(\mathbb{Q}, d)$, in fact an ultrametric space because it satisfies the strong Δ-~~ineq~~ inequality: (4)

$d(\alpha,\beta) \leq \max(d(\alpha,\beta), d(\gamma,\beta))$

Exercise: Prove that if $d(\alpha,\beta) \neq d(\gamma,\beta)$ (resp. $|\alpha|_p \neq |\beta|_p$ in 3 above) then in (resp. in 3 above) ~~we have~~ one has $=$.

But we will, in the proof, use the norm notation $|\alpha|_p$, not $d(\cdot,\cdot)$.

The above prop. says that $(\mathbb{Q}, +, \cdot, |\cdot|_p)$ is a normed non-Archimedean field. There is the discipline of p-adic Analysis [A.M. Robert, A Course in p-adic Analysis, Springer, 2000] which is Mathematical Analysis in $(\mathbb{Q}, |\cdot|_p)$ (actually in $(\mathbb{Q}_p, |\cdot|_p)$ or $(\mathbb{C}_p, |\cdot|_p)$ where $\mathbb{Q}_p$ is the p-adic $\mathbb{R}$ and $\mathbb{C}_p$ the p-adic $\mathbb{C}$). These ~~are~~ analytical methods in NT of sorts too, are, so this is not as big digression as one might ~~believe~~ think.

But back to the proof of the von S.-C. Thm.
We will use this corollary of (3): $\forall \alpha_1, \alpha_2, \ldots, \alpha_5 \in \mathbb{Q}$;

(5)

$|\alpha_1+\alpha_2+\dots+\alpha_n|_p \le \cancel{\max(|\alpha_1|, \dots)}$
$\le \max(|\alpha_1|_p, |\alpha_2|_p, \dots, |\alpha_n|_p)$. For example, if ~~$|\alpha_n|_p$~~ $|\alpha_n|_p \le 1 \ \forall n \in \mathbb{N}$, then $|\alpha_1+\alpha_2+\dots+\alpha_n|_p \le 1 \ \forall n \in \mathbb{N}$. Of course, compared to the usual ~~p~~ Euclidean norm $|\cdot|$ on $\mathbb{R}$ or on $\mathbb{C}$, this is quite counter-intuitive. <u>Advantage</u>: in p-adic Analysis, a series $\sum_{n=1}^{\infty} a_n$ converges (in $|\cdot|_p$) $\Longleftrightarrow$ $\lim_{n\to\infty} a_n = 0$ (in $|\cdot|_p$).

If only we had this in the normal Analysis in $(\mathbb{R}, |\cdot|)$ and $(\mathbb{C}, |\cdot|)$. <u>Disadvantage</u>: there is nothing like connected interval in p-adic Analysis, in fact p-adic spaces are totally disconnected (all components are singletons).

Claim 1 $\lim_{n\to 0} \frac{1}{n} S_k(n) = B_k$ (in $|\cdot|_p$).
$k \in \mathbb{N}$ ~~even~~:

Proof. Here it is understood that n runs through a sequence of integers which go to 0 in the p-adic norm $|\cdot|_p$, for example $n = p, p^2, p^3, p^4, \ldots, p^m, \ldots$ From Prop. B we get:

$$\left|\frac{1}{n} S_k(n) - B_k\right|_p = \left| \sum_{r=0}^{k-1} \underbrace{\binom{k}{r} \frac{B_r}{k+1-r}}_{c_r} n^{k-r} \right|_p \leq$$

$$\leq \max_{0 \leq r \leq k-1} \left(|c_r|_p \cdot |n|_p^{k-r} \right) \leq \underbrace{|n|_p}_{\to 0} \cdot \max |c_r|_p \to 0. \quad \boxtimes$$

(in $|\cdot|$, usual Eucl. norm)

Claim 2 If p is a prime number and $k \in \mathbb{N}$, then

$$S_k(p) = \sum_{j=0}^{p-1} j^k \equiv \begin{cases} -1 & (p-1) \mid k \\ 0 & \text{else} \end{cases}$$

modulo p.

Proof. ~~By~~ Little Fermat's thm. says that $j^{p-1} \equiv 1 \pmod p$ for every $j = 1, 2, \ldots, p-1$, which gives the first case. I will continue with the 2nd case next week on May 1st and I mean it!