(L7) (April 17, 2020) | So let us return to a proof for a generalization of the well known formula 1

(*) $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ By Wikipedia, the problem to sum the series $\sum \frac{1}{n^2}$ was posed by P. Mengoli in 1650 and solved by L. Euler in 1734. It may be of some interest to recall here Euler's original solution.

If $p \in \mathbb{C}[x]$ with $p(0) = 1$, for example $p_0(x) = -2x^2+1$, we can factorize $p(x)$ into linear polynomials as $p(x) = \prod_{j=1}^{n} (1 - \frac{x}{\alpha_j})$ where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the roots of $p(x)$, so $p_0(x) = (1 - \frac{x}{1/\sqrt{2}})(1 + \frac{1}{1/\sqrt{2}})$. Multiplying the product out and comparing coefficients (of x^2) on both sides we get identities like

a_1 = the coeff. of x in $p(x) = -(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \dots + \frac{1}{\alpha_n})$.

L. Euler did this for the infinite „polynomial" $p_1(x) := \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots$

Clearly, $p_1(0) = \lim_{x\to 0} \frac{\sin x}{x} = 1$ and the "roots" of $p_1(x)$ are the numbers $\pm\pi, \pm 2\pi, \ldots$

Thus $p_1(x) = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right)$, $\left(= \left(1-\frac{x}{\pi}\right)\left(1+\frac{x}{\pi}\right)\left(1-\frac{x}{2\pi}\right)\left(1+\frac{x}{2\pi}\right)\ldots\right)$ and equating the coeff-s of x^2 on both sides of ~~[illegible]~~ gives the identity

$$-\frac{1}{3!} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2},$$ which is (*).

Of course, many objections can be rised to Euler's solution and one of them, stated already at the time, is how do we know that the "polynomial" $p_1(x) = \frac{\sin x}{x}$ does not have some other (complex) roots besides $\pm\pi, \pm 2\pi, \ldots$

The on-line available text "Evaluating $\zeta(2)$" by R. Chapman gives 14 proofs of Euler's identity (*). With some effort one can make Euler's "proof" rigorous but in these lecture notes we will follow another path. ~~But~~ First we have to introduce so called <u>Bernoulli numbers B_n</u>. One of possible definitions is that those are the coeff-s in the power series

$$\boxed{\frac{x}{e^x-1} =: \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}} \quad (B)$$

Clearly, $B_n \in \mathbb{Q}$ because $\sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = -x \sum_{q=0}^{\infty} (e^x)^q$ — ha, this actually is not well defined in the ring $\mathbb{C}[[x]]$ of formal power series, <u>an (exercise) for you (what's the problem here?)</u> — (2nd attempt) =

$$= \frac{x}{\sum_{q=1}^{\infty} \frac{x^q}{q!}} = \frac{1}{\sum_{q=1}^{\infty} \frac{x^{q-1}}{q!}} = \frac{1}{1+\frac{x}{2!}+\frac{x^2}{3!}+\dots} =$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2!}+\frac{x^2}{3!}+\dots\right)^n \quad \left[\frac{1}{1+y} = 1-y+y^2-y^3+\dots\right]$$

and so indeed $B_n \in \mathbb{Q}$.

(Exercise) Prove that $B_0 = 1$, $B_1 = -\frac{1}{2}$ but $B_{2n+1} = 0$ for every $n = 1, 2, 3, \dots$

The first few initial values of B_n are:

n	0	1	2	4	6	8	10	12	14	16
B_n	1	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{30}$	$\frac{1}{42}$	$-\frac{1}{30}$	$\frac{5}{66}$	$-\frac{691}{2730}$	$\frac{7}{6}$	$-\frac{3617}{510}$

Exercise: Deduce from (B) that

$\limsup_{n\to\infty} |B_n|^{1/n} = +\infty$, i.e. $|B_n|$ grows superexponentially (which is not evident from the table).

B. numbers are a huge and fascinating subject in itself and a whole course could be given only on them (I did something like it many years ago) but besides their role in generalizing (*), here I only give one (or two...) of their spectacular successes. First, their relation to the FLT (Fermat's last theorem). In 1850 E.E. Kummer (1810–1893) proved: If $p > 2$ is a prime number and p does not divide any of the numerators of the B. numbers $B_2, B_4, B_6, \ldots, B_{p-3}$ then $x^p + y^p + z^p = 0$ has no solution $x, y, z \in \mathbb{Z} \setminus \{0\}$. So, for example, FLT holds for the exponent $p=13$ because B_2, B_4, B_6, B_8 and B_{10} have respective numerators $1, -1, 1, -1, 5$ – none is divisible by 13. However, for many larger p this criterion fails, for example for $p=691$ or for $p=37$ which divides the numerator of B_{32}. Still I cannot resist and ~~to present~~ present and prove one more nice ~~result on~~ property of Bernoulli numbers:

Theorem (Von Staudt – Clausen)

$$B_{2n} + \sum_{\substack{p \text{ prime} \\ (p-1)|2n}} \frac{1}{p} \in \mathbb{Z}.$$

For example: $B_2 = \frac{1}{6}$ and the only primes p s.t. $p-1$ divides 2 are $p=2$ and $p=3$, and indeed $\frac{1}{6}+\frac{1}{2}+\frac{1}{3} = \frac{1}{6}+\frac{5}{6} = 1$. Or, ~~for~~ $B_{12} = -\frac{691}{2730}$, 12 has divisors $1, 2, 3, 4, 6$ and 12, increased by 1 they are ②, ③, 4, ⑤, ⑦ and

(13) – primes are circled – and indeed

$$-\frac{691}{2730}+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{13}=$$

$$=(-\|-)+\frac{3\cdot5\cdot7\cdot13+2\cdot5\cdot7\cdot13+2\cdot3\cdot7\cdot13+2\cdot3\cdot5\cdot13+2\cdot3\cdot5\cdot7}{2\cdot3\cdot5\cdot7\cdot13}=$$

$$=(-\|-)+\frac{1365+910+546+390+210}{30\cdot91}=$$

$$=\frac{-691}{2730}+\frac{2275+1146}{2730}=\frac{3421-691}{2730}$$

$$=\frac{2730}{2730}=1.$$ I will prove the theorem next time, that is on Friday, April 24.