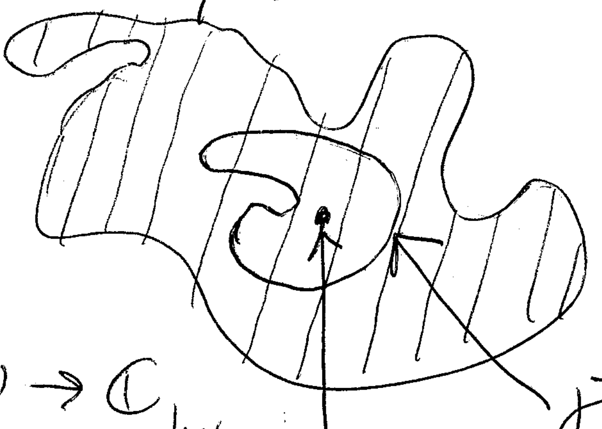
(L6) (April 3, 2020) I concluded the last lecture with the precise (well, up to the definition of the interior of a closed simple and piecewise smooth curve) definition of the Cauchy formula $f(z_0) = \int_\gamma \frac{f(z)\,dz}{z - z_0}$.



U:

$f: U \to \mathbb{C}$ holom.,

$\gamma: [a,b] \to U$.

z_0

$\gamma([a,b])$

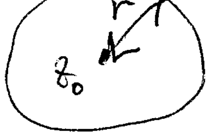
First I prove, or rather derive, by means of this formula a generalization of the identity $\boxed{\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}}$. This identity, as well as the generalization, is (I think) due to L. Euler. For the generalization we have first to state a corollary of the C. formula, the theorem on residues, and then to recall the Bernoulli numbers.

I will call a simple closed piecewise smooth curve $\gamma: [a,b] \to \mathbb{C}$ simply an admissible curve: the C. formula follows from the theorem (A.L. Cauchy, c. 1830).

If $f: U \to \mathbb{C}$ is a holomorphic function ($U \subset \mathbb{C}$ is $\neq \emptyset$ and open), $\gamma: [a,b] \to U$ is an adm. curve and the interior of $\gamma \subset U$, then $\int_\gamma f = 0$. This theorem in fact is properly called the Cauchy-Goursat theorem because Cauchy always assumed that f' is continuous, and only Goursat proved the theorem in 1900 (an article in French in the Bulletin of Amer. Math. Soc.) in the complete generality, only with the assumption that f' exists.

For a proof of the Cauchy theorem (and also of the C. formula) see my lecture notes Mathematical Analysis 3 (→ link on the web page of this course, both English & Czech version).

For a $z_0 \in \mathbb{C}$ and a real $r > 0$ let $D(z_0, r) = \{z \in \mathbb{C} \mid |z - z_0| < r\}$ be the (open) disc



centered at z_0

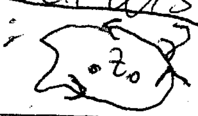
...ne with radius r.

Proposition (Complement to the C. theorem)

If $\gamma : [a,b] \to \mathbb{C}$ is a closed curve$^{(+)}$ and $n \in \mathbb{Z}$ is an integer then

$$\int_\gamma (z-z_0)^n \, dz = \begin{cases} 2\pi i & \dots \ n = -1 \\ 0 & \dots \ n \neq -1. \end{cases}$$

(+) going counter-clockwise around z_0:



For $n<0$ the ~~fn~~ integrand is holomorphic on the punctured plane $\mathbb{C} \setminus \{z_0\}$. We ~~give~~ mention two more fundamental properties of holomorphic functions.

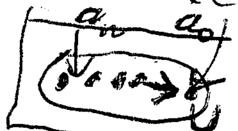
Theorem (local power series expansions) Let $U \subset \mathbb{C}$ be a $\neq \emptyset$ open set, $f : U \to \mathbb{C}$ be a holomorphic function and $D = D(z_0, r) \subset U$ be a disc contained in U. Then there exist coefficients $a_n \in \mathbb{C}$, $n \in \mathbb{N}_0$, s.t.

$$\forall z \in D : f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n,$$

i.e. f is the sum of a power series centered at z_0.

Theorem (rigidity of holom. functions) Let $U \subset \mathbb{C}$ be a $\neq \emptyset$, open and connected set and $f, g : U \to \mathbb{C}$ be holom. functions.

Further let $(a_n)=(a_1,a_2,\ldots)\subset U$ be a sequence of points in U s.t. $\lim_{n\to\infty} a_n = a_0\in U$ but for every $n\in\mathbb{N}$ we have that $a_n\neq a_0$. Then



$$\big(\forall n\in\mathbb{N}: f(a_n)=g(a_n)\big)\Longrightarrow\big(\forall z\in U: f(z)=g(z)\big).$$

~~Exercise~~ First, recall that an open set $U\subset\mathbb{C}$ is connected if there is no partition $U=V\cup W$ where $W, V\neq\emptyset=V\cap W$ and both V and W are open too.

Exercise: Every $\neq 0$ holom. function $f:U\to\mathbb{C}$, where U is $\neq\emptyset$, connected and open, has discrete zero set $Z(f)=\{z_0\in U\mid f(z_0)=0\}$

$$\forall z_0\in Z(f)\ \exists r>0: D(z_0,r)\cap Z(f)=\{z_0\}.$$

Definition: Let $U\subset\mathbb{C}$ be ~~$\neq\emptyset$~~ and open. We say that a function $f:U\setminus X\to\mathbb{C}$ is meromorphic (on U), where $X\subset U$ is a discrete subset of U, if there exist holom. functions $g,h:U\to\mathbb{C}$ s.t. $Z(h)=\{z_0\in U\mid h(z_0)=0\}=X$ and $f=\frac{g}{h}$.

on $U \setminus X$. For example $f(z) = e^z + \frac{1}{(z-1)^3} =$

$= \frac{e^z(z-1)^3+1}{(z-1)^3}$ is meromorphic on $\mathbb{C}\setminus\{1\}$.

Or $\frac{1}{\sin(z)}$ is meromorphic on $\mathbb{C}\setminus\pi\mathbb{Z}$.

Exercise (*) Prove that if $f: U\setminus X \to \mathbb{C}$ is merom. then for $\forall a \in X$ $\exists r_a > 0$ $\exists m_a \in \mathbb{Z}$ $\exists c_{a,m_a}, c_{a,m_a+1}, \ldots \in \mathbb{C}$ s.t. $D(a,r_a) \subset U$ & $D(a,r_a) \cap X = \{a\}$ & $z \in D(a,r_a) \setminus \{a\} \Rightarrow f(z) = \sum_{n=m_a}^{\infty} c_{a,n}(z-a)^n$.

The coefficient $c_{a,-1}$ is so called residue of f at a. denoted $\boxed{\mathrm{Res}(f,a)}$

Finally we arrive at the

Theorem (on residues) (γ is counter-clockwise oriented!)

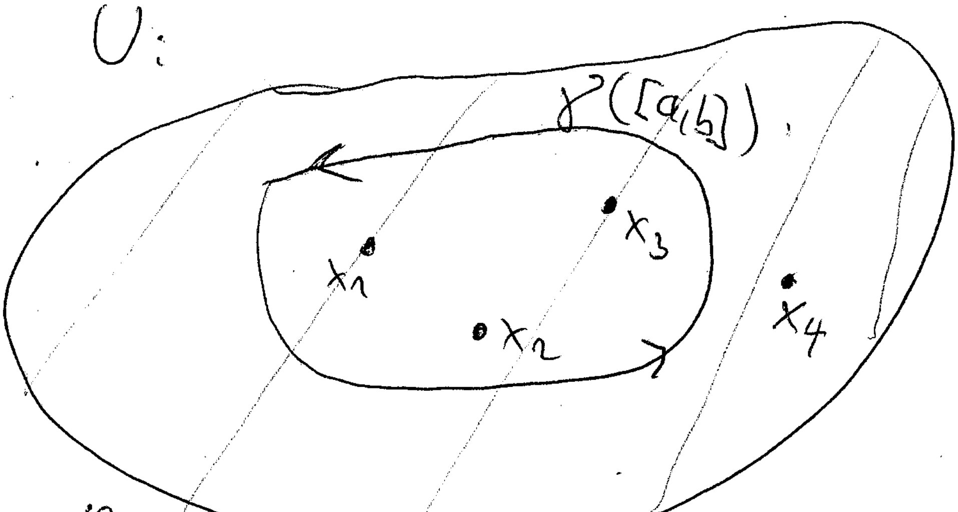
Let $f: U\setminus X$ be a merom. function, $\gamma: [a,b] \to U$ be an adm. curve s.t. $\underbrace{\text{the interior of } \gamma}_{=:\mathrm{Int}(\gamma)} \subset U$ and $\gamma([a,b]) \cap X = \emptyset$. Then

$$\boxed{\int_\gamma f = \sum_{a \in X \cap \mathrm{Int}(\gamma)} 2\pi i \cdot \mathrm{Res}(f,a).}$$

Exercise Prove that the sum ~~of the~~ is finite.

Proof. The theorem follows from the Exercise (*), the previous Proposition, ~~and~~ the fact that $\sum_{n\geq 0} a_n (z-z_0)^n$ is holom. on the disc of convergence $D(z_0, r)$, and the C. theorem.

U:

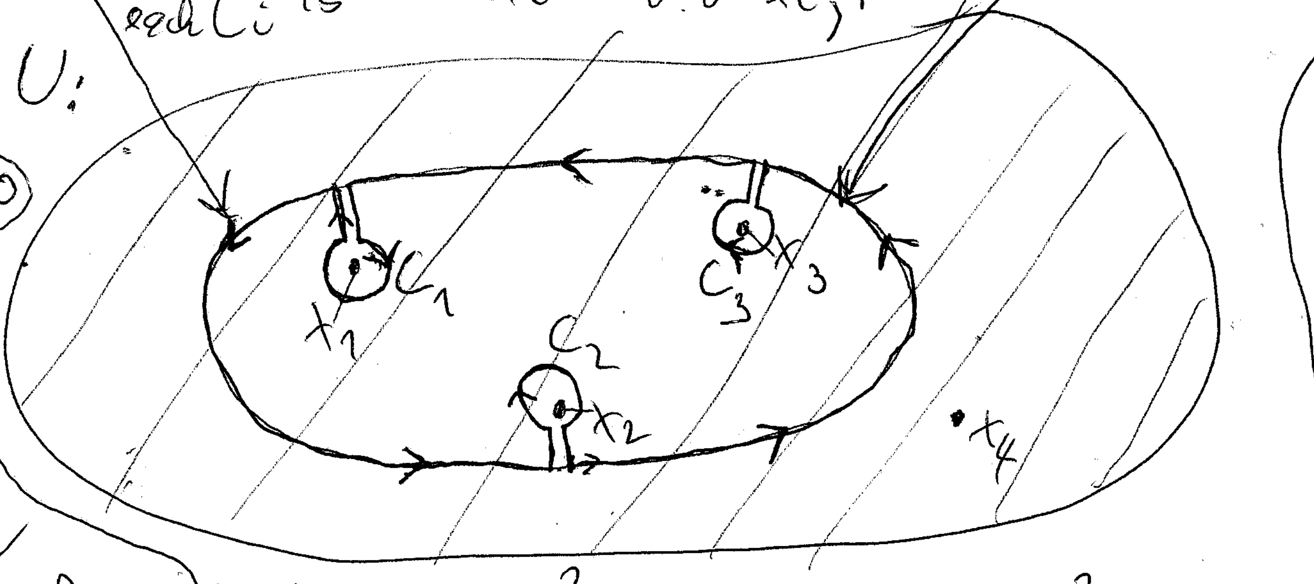


$X = \{x_1, x_2, x_3, x_4\}$

We replace the curve γ with δ:

small clockwise for $x_i \in \mathrm{Int}(\gamma)$, each C_i is a circle around x_i; an adm. curve $\delta([a,b])$

U:



$\int f \doteq 0$

Then

$$0 = \int_\delta f \doteq \int_\gamma f + \sum_{i=1}^{3} \int_{C_i} f \doteq \int_\gamma f + \sum_{i=1}^{3} (-2\pi i \cdot \mathrm{Res}(f, x_i))$$

f is holom. on $\mathrm{Int}(\delta)$; we apply the Cauchy thm. | additivity of $\int$ | C_i goes clockwise around x_i and by $\boxtimes$