(L4) (March 20, 2020(!)) Contrary to what I wrote last time, I will prove Mertens' 1st formula:

Theorem (F. Mertens, 1874) $\forall x \geq 2$ one has that
$$\sum_{p\leq x} \frac{\log p}{p} = \log x + O(1).$$

However, for the ~~complete~~ proof we need ~~to prove~~ the so called Chebyshev's (Čebyšev, Чебышёв) estimates:

Theorem (P. L. Čebyšev, ≐ 1850) Let, for real x, $\pi(x) = \#\{p \leq x \mid p$ is a prime number$\}$ — the number of primes not exceeding x. Then, for some real constants $0 < c_1 < c_2$ and every real $x \geq 2$,
$$\boxed{\frac{c_1 x}{\log x} < \pi(x) < \frac{c_2 x}{\log x}}$$

For the proof of see my LN Introduction to Number Theory (link on the webpage), where in fact the 1st M. formula is proven as well.

Proof (of 1st Mertens thm.). We use the von Mangoldt function $\Lambda: \mathbb{N} \to \mathbb{R}$, $\Lambda(n) = \begin{cases} \log p & \dots\ n = p^a \\ 0 & \dots\ n \neq p^a. \end{cases}$

First we prove that for $x \geq 2$,
$$\sum_{n\leq x} \Lambda(n)\left\lfloor \frac{x}{n} \right\rfloor = \log(\lfloor x \rfloor !) = x\log x + O(x).$$

Indeed, $\sum_{n\le x}\Lambda(n)\lfloor\frac{x}{n}\rfloor = \sum_{n\le x}\Lambda(n)\sum_{\substack{m,\, m\le x\\ n\mid m}} 1 = (\Sigma\Sigma)$ (n divides m)

$$= \sum_{m\le x}\sum_{\substack{n\\ n\mid m}}\Lambda(n) = \sum_{m\le x}\log m = (\log(\lfloor x\rfloor !)) =$$

$= x\log x + O(x)$ (in L2 we proved an even stronger estimate of the sum). The equality follows from the prime factorization $m = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k}$ — we note that the sum is, by the def. of Λ, $a_1\log p_1 + \cdots + a_k\log p_k = \log m$.

Thus $x\log x + O(x) =$

$$= \sum_{n\le x}\Lambda(n)\lfloor\frac{x}{n}\rfloor = x\sum_{p\le x}\frac{\log p}{p} + O(x) + \sum_{\substack{p,\, r\ge 2\\ p^r\le x}}\log p\cdot\lfloor\frac{x}{p^r}\rfloor$$

this follows from the def. of Λ and from writing $\lfloor\frac{x}{p}\rfloor = \frac{x}{p} - \{\frac{x}{p}\}$, $\{\frac{x}{p}\}\in[0,1)$ and estimating the sum

$$\left|\sum_{p\le x}\Lambda(p)\{\frac{x}{p}\}\right| = \left|\sum_{p\le x}\log p\cdot\{\frac{x}{p}\}\right| \le$$

$$\le \sum_{p\le x}\log p \le (\log x)\pi(x) \ll \log x\cdot\frac{x}{\log x} = x.$$

Here $\ll$ is the same as $O(\cdot)$ and is the upper

Čebyšev's bound. We estimate the sum (...) : ③

$$(\ldots) \le x \sum_{n,r \ge 2} \frac{\log n}{n^r} \le x\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots\right)\sum_{n=2}^{\infty} \frac{\log n}{n^2}$$

$= O(x)$.

because the series ~~converges~~.

Now if we divide the equality .... by x, we get that indeed

$$\log x + O(1) = \sum_{p \le x} \frac{\log p}{p} + O(1). \quad \boxtimes$$

Stieltjes integral(s) - named after the Holland mathematician <u>Thomas Stieltjes</u> (1856-1894) - is used in analytic NT and generalizes the Riemann $\int$. Its main advantage is that in this integral the difference between $\sum$s and $\int$s disappears. Here we only give its definition and basic identities.

Definition

Let $f, g : [a,b] \to \mathbb{R}$ be functions. The Stieltjes $\int$ of f over $[a,b]$ with respect to g is the limit

$$L = \lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(\xi_i)\cdot(g(t_i) - g(t_{i-1}))$$ if it exists

where $\Delta = (t_0, t_1, \ldots, t_n)$, $a = t_0 <$

$< t_1 < t_2 < \dots < t_{n-1} < t_n = b$ is a division of the interval $[a,b]$, $\|\Delta\| = \max_{1\le j\le n}(t_j - t_{j-1})$ and $\xi_j \in [t_{j-1}, t_j]$

We write $\int_a^b f\,dg := L$ | The Riemann $\int$ is a particular case with $g(x)=x$.

The main existential theorem for the S. integral is: $\boxed{\text{Theorem}}$ If $f, g : [a,b] \to \mathbb{R}$ ($a<b$ are real numbers), f is continuous and g is of bounded variation, then $\int_a^b f\,dg$ exists.

Recall that $\mathrm{Var}_{[a,b]}(g) = \sup\left(\left\{\sum_{j=1}^{n} |g(t_j) - g(t_{j-1})| \;\middle|\; a = t_0 < t_1 < \dots < t_n = b \text{ is a division of } [a,b]\right\}\right)$.

$\boxed{\text{Theorem}}$ 1) If $\int_a^b f\,dg$ exists then $\int_a^b g\,df$ exists and $\int_a^b g\,df = f(b)g(b) - f(a)g(a) - \int_a^b f\,dg$.

2) If $g \in C^1(a,b)$ and $(R)\int_a^b f$ exists then $\int_a^b f\,dg = \int_a^b f(x)g'(x)\,dx = (R)\int_a^b f g'$. Also, $\mathrm{Var}_{[a,b]}(g) = (R)\int_a^b |g'|$.

For example, ~~it~~ with

By S.-S. we can write, for $A \subset \mathbb{N}$, $x > 1$, and $f \in C(1, x)$, that – denoting as before $A(x) = \sum_{\substack{n \le x \\ n \in A}} 1 = \#\{n \le x \mid n \in A\}$ –

esterciso

$$\int_{1-0}^{x} f(t)\, dA(t) = \cdots = \sum_{\substack{n \le x \\ n \in A}} f(n).$$

We return to theorems on Σs and $\int$s. ((R) $\int$s)

Theorem 4 (Euler–MacLaurin, general form)

Let $n \in \mathbb{N}$, $a < b$ be ~~real numbers~~ integers, $f \in C^{2n}(a,b)$. Then

$$\sum_{k=a}^{b} f(k) = \int_a^b f + \frac{f(a)+f(b)}{2} + \sum_{k=1}^{n} \left[\frac{B_{2k}}{(2k)!} f^{(2k)} \right]_a^b -$$

$$- \frac{1}{(2n)!} \int_a^b B_{2n}(t - \lfloor t \rfloor) f^{(2n)}(t)\, dt, \text{ where}$$

the B_i are so called Bernoulli numbers, defined implicitly by $\frac{y}{e^y - 1} = \sum_{i=0}^{\infty} B_i \frac{y^i}{i!}$ and $B_i(x)$ is the i-th B.-polynomial.

The Bernoulli polynomial $B_i(x)$ is the unique polynomial of degree i s.t. $\int_x^{x+1} B_i(u)\, du = x^i$. We have

$B_1(u) = u - \frac{1}{2}$, $B_2(u) = u^2 - u + \frac{1}{6}$, $B_3(u) = u^3 - \frac{3}{2}u^2 + \frac{1}{2}u, \ldots$; As for the B. numbers, $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_2 = 1/6$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = -\frac{1}{30}$,