$\boxed{\text{L 12, May 22, 2020}}$ It remains to prove (1)
Prop. 5 and Thm. 6 of the previous lecture. For the former we need

Theorem 7 $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \neq 0$ for $s \in \mathbb{C} \setminus \{1\}$ with $\mathrm{Re}(s) \geq 1$ (we know ~~that~~ $\zeta(s)$ is holom. on (from L 10)).

Proof. For $\mathrm{Re}(s) > 1$, we have the identity

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \cdot \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = 1$$

where μ is the Möbius function(*) ($\sum_{d|n} \mu(d) = \begin{cases} 1 & \ldots n=1 \\ 0 & \ldots n>1 \end{cases}$), so $\zeta(s) \neq 0$ there.

The hard part is to prove that $\zeta(s) \neq 0$ if $\mathrm{Re}(s) = 1$, $s \neq 1$. For $\mathrm{Re}(s) > 1$ we have another identity, the

$$\boxed{\text{Euler product } \zeta(s) = \prod_p \frac{1}{1-p^{-s}} \text{ (Exercise)}}$$

So $\log(|\zeta(s)|) = \mathrm{Re}\left(\sum_p \log(1-p^{-s})^{-1}\right)$, $\mathrm{Re}(s) > 1$

$$= \mathrm{Re}\left(\sum_p \left(p^{-s} + \frac{p^{-2s}}{2} + \frac{p^{-3s}}{3} + \ldots\right)\right) = \sum_{n=1}^{\infty} a_n n^{-\sigma} \cdot \cos(t \log n)$$

where $s = \sigma + it$, $\sigma > 1$, and

$$a_n = \begin{cases} \frac{1}{r} & \ldots n = p^r \\ 0 & \ldots \text{else.} \end{cases}$$

We have $a_n \geq 0$ and $n^{-\sigma} \geq 0$ but have to tame the sign changes of the $\cos(\ldots)$. We use the trigonometric

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(*) $\mu : \mathbb{N} \to \{-1, 0, 1\}$, $\mu(n) = \begin{cases} (-1)^r & \ldots n = p_1 p_2 \cdots p_r,\ p_i \text{ distinct prime numbers} \\ 0 & \ldots \text{else} \end{cases}$

identity that $\forall x \in \mathbb{R}$: $3+4\cos x+\cos(2x) = 2(1+\cos x)^2 \geq 0$. Then $\forall \sigma, t \in \mathbb{R}, \sigma > 1$:

$$\boxed{\log\left(|\zeta(\sigma)^3 \zeta(\sigma+it)^4 \zeta(\sigma+2it)|\right) =}$$
$$\boxed{= \sum_{n=1}^{\infty} 2 a_n n^{-\sigma} (1+\cos(t\log n))^2 \geq 0.}$$

This inequality implies that $\zeta(1+it) \neq 0$ for $\forall t \in \mathbb{R}$. For contradiction, let $\zeta(1+it_0)=0$ for some $t_0 \in \mathbb{R}\setminus\{0\}$. Then for $\sigma \to 1^+$,

$$\zeta(\sigma)^3 \zeta(\sigma+it_0)^4 \zeta(\sigma+2it_0) =$$

$= O(\sigma-1)$, because $\cdot \sim (\sigma-1)^{-3}$ ($\zeta(s) - \frac{1}{s-1}$ is entire), $\cdot = O((\sigma-1)^4)$ (local expansion of ζ around the zero $1+it_0$) and $\cdot = O(1)$ (trivi)

Thus for $t=t_0$ and $\sigma \to 1^+$, in $\boxed{\ldots}$ we have that $\boxed{\ldots} \to -\infty$, which is a contradiction. ☒

$\boxed{\text{Proof of Prop. 5}}$ We extend $F(s) - \frac{1}{s-1} = \sum_p \frac{\log p}{p^s} - \frac{1}{s-1}$ holom-aly to $Re(s) \geq 1$. This provides a holom. extension of $\frac{F(z+1)}{z+1} - \frac{1}{z} = \frac{1}{z+1}\left(F(z+1) - \frac{1}{z} - 1\right)$ to $Re(z) \geq 0$. For $Re(s) > 1$ by taking $(\log(\ldots))'$ of the Euler product for $\zeta(s)$ we get

$$\frac{\zeta(s)'}{\zeta(s)} = (\log(\zeta(s)))' = \cdots = -\sum_p \frac{\log p}{p^s - 1} =$$

$$= -F(s) - \sum_p \frac{\log p}{p^s(p^s-1)}.$$ Hence, for $Re(s) > 1$,

$$\boxed{F(s) - \frac{1}{s-1} = -\left(\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}\right) - \sum_p \frac{\log p}{p^s(p^s-1)}.}$$

We claim that the right side is holom. on $Re(s) \geq 1$. The $\sum \cdots$ is clearly holom. for $Re(s) > \frac{1}{2}$, so the crucial term is $\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} =$ [writing $\zeta(s) = \frac{1}{s-1} + z(s)$ with an entire - holom. on $\mathbb{C}$ - $z(s)$] $=$

$$= \frac{-(s-1)^{-2} + z'(s)}{(s-1)^{-1} + z(s)} + \frac{1}{s-1} = \frac{z(s) + (s-1)z'(s)}{1 + (s-1)z(s)}.$$

This expression shows that $\boxed{\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}}$ is holom. on a neighborhood of $s = 1$. Holomorphicity of $\cdot$ on a neighborhood of $s = 1 + it$, $t \neq 0$, follows from Thm. 7 which shows that there the denominator $\zeta(s) \neq 0$. $\boxtimes$

Proof of Theorem 6 (see the last lecture) For a bounded function $f: [0, +\infty) \to \mathbb{R}$, integrable on bounded intervals, we take its Laplace transform $g(z) := \int_0^{+\infty} f(t) e^{-tz}\, dt$, $Re(z) > 0$, $(z \in \mathbb{C})$ and

assume that $g(z)$ has a holom. extension to $\mathrm{Re}(z) \geq 0$.
We then show that $\boxed{\int_0^{+\infty} f(t)\,dt = g(0).}$

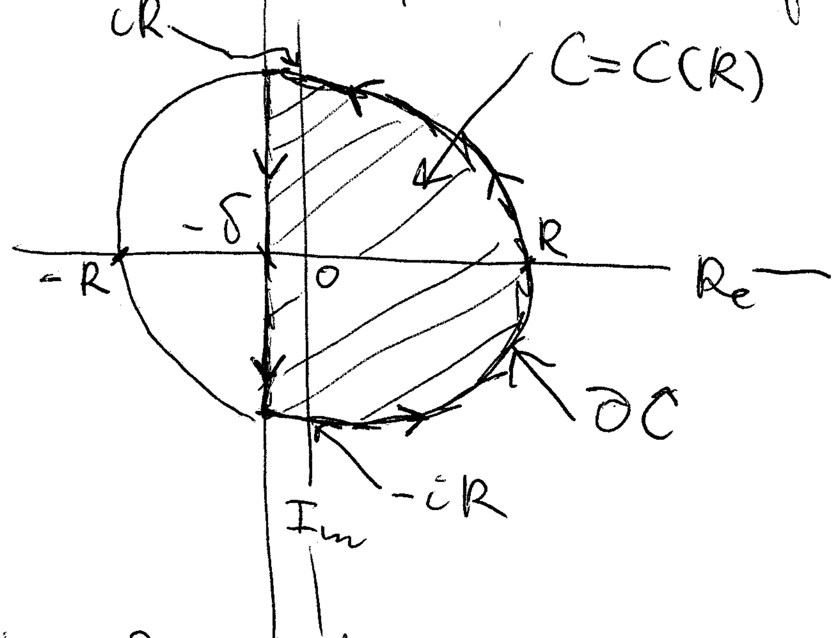
For real $T > 0$ let $g_T(z) := \int_0^T f(t) e^{-tz}\,dt$ — this is an entire function (holom. on the whole $\mathbb{C}$). We prove that

$$\boxed{\lim_{T\to+\infty} g_T(0) = g(0).}$$

Let $R > 0$ be real and

$\underset{C(R)}{C} := \{ z \in \mathbb{C} \mid |z| < R \ \&\ \mathrm{Re}(z) > -\delta \}$

where $\delta = \delta(R) > 0$ is small enough that $g(z)$ extends holom-ally to the closure $\bar{C}$ (such δ exists because the segment $[-iR, iR]$ is compact):



By the Cauchy theorem,

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_{\partial C} \frac{g(z) - g_T(z)}{z}\,dz$$

where ∂C is the ↺-oriented boundary contour of C. Newman's trick is to introduce in a kernel $G(z)$:

$$g(0) - g_T(0) = \frac{1}{2\pi i} \int_{\partial C} \frac{g(z) - g_T(z)}{z}\, G(z)\,dz,$$ for any function $G(z)$ that is

holom. on $\overline{C}$ and has value $G(0)=1$. We set ⑤

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$$G(z)=G(z,R,T)=\left(1+\frac{z^2}{R^2}\right)e^{zT}$$ where $R,T>0$ and have the previous meanings. $G(z)$ is entire and $G(0)=1$. If $|z|=R$ then

$$(\ast)\quad \left|\frac{G(z)}{z}\right|=\left|\frac{e^{zT}(z+\bar{z})}{R^2}\right|=$$

$$=2e^{\operatorname{Re}(z)T}\frac{|\operatorname{Re}(z)|}{R^2}.$$ It suffices to show that

$$\boxed{I:=\int_{\partial C}\frac{g(z)-g_T(z)}{z}G(z)\,dz\to 0 \text{ as } T\to+\infty.}$$

We split I in three integrals: if ∂C^- (resp. ∂C^+) is $\partial C\cap(\operatorname{Re}(z)\le 0)$ (resp. $\partial C\cap(\operatorname{Re}(z)>0)$) and $K^-=$
$=(|z|=R)\cap(\operatorname{Re}(z)\le 0)$, then

$$I=\int_{\partial C^-}\frac{g(z)}{z}G(z)\,dz-\int_{\partial C^-}\frac{g_T(z)}{z}G(z)\,dz+\int_{\partial C^+}\frac{g(z)-g_T(z)}{z}G(z)\,dz$$

$$=\text{—}\,\text{''}\,\text{—}-\int_{K^-}\frac{g_T(z)}{z}G(z)\,dz+\text{—}\,\text{''}\,\text{—}$$

$=:I_1-I_2+I_3$. In I_2 we could replace ∂C^- with K^- because the integrand is holom. on $\mathbb{C}\setminus\{0\}$ (we add an $\int\ldots=0$). ⑤

To bound $|I_1|$, we write it as $I_1=\int_{\partial C^-}J(z)e^{zT}dz$ where $J(z)=$

$= \frac{g(z)}{z}(1+z^2R^{-2})$ does not depend on T. Let $r_1 = r_1(R) :=$
$:= \max_{z\in\partial C^-} |J(z)|$. Then $|I_1| \le r_1 \int_{\partial C^-} |e^{zT}|\, dz$. $\forall \varepsilon>0\ \exists \varkappa>0$:
$|e^{zT}| \le e^{-\varkappa T}$ for every $z\in\partial C^-$, except for a part of ∂C^- with length $\le \varepsilon \cdot$ length of ∂C^-, which lies close to the imaginary axis – there we use the trivial estimate $|e^{zT}|\le 1$.

Thus $|I_1| \le r_1(e^{-\varkappa T}+\varepsilon)\underbrace{|\partial C^-|}_{\text{the length of } \partial C^-} < 3 r_1 R(e^{-\varkappa T}+\varepsilon)$

$\rightsquigarrow$ $\boxed{\forall \text{ fixed } R>0: \ \lim_{T\to+\infty}|I_1| = 0.}$

Integrals I_2 and I_3 are estimated with the help of the kernel $G(z)$. Let $B := \sup_{t\ge 0}|f(t)|$. For $\mathrm{Re}(z)<0$ we have:

$$|g_T(z)| = \left|\int_0^T f(t)e^{-tz}\,dt\right| \le B\int_{-\infty}^T |e^{-tz}|\,dt = \frac{B e^{-\mathrm{Re}(z)T}}{|\mathrm{Re}(z)|}.$$

For $\mathrm{Re}(z)>0$ similarly

$$|g(z)-g_T(z)| \le B\int_T^{+\infty}|e^{-tz}|\,dt = \frac{B e^{-\mathrm{Re}(z)T}}{\mathrm{Re}(z)}.$$

Using the above expression (0) for $\left|\frac{G(z)}{z}\right|$ on $|z|=R$ and the lengths $|K^-| = |\partial C^+| = \pi R$, we get the next estimates for I_2 and I_3:

$|I_2| \leq \frac{2\pi B}{R}$ and $|I_3| \leq \frac{2\pi B}{R}$ which are independent of T. To conclude, for given $\varepsilon > 0$ we fix an $R > 0$ s.t. $R > \frac{8\pi B}{\varepsilon}$ and fix the corresponding domain $C = C(R)$. Then $|I_2| + |I_3| < \frac{\varepsilon}{2}$ for every $T > 0$. We know that for that fixed R:

$T > T_0 \Rightarrow |I_1| < \frac{\varepsilon}{2}$. Hence

$$T > T_0 \Rightarrow |I| \leq |I_1| + |I_2| + |I_3| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We have proven that $\lim_{T \to +\infty} I = 0$. This finishes the proof of the PNT. $\boxtimes$

Thank you for your attention and patience.