L 11, 15 May, 2020 | The Prime Number

$\boxed{\text{Theorem (PNT)}}$ $\pi(x) \sim \frac{x}{\log x}$, i.e.

$$\lim_{x\to+\infty} \frac{\pi(x)}{x/\log x} = 1$$ where $\pi(x) = |\mathbb{P} \cap (-\infty, x]|$ is the number of primes $\leq x$, $\mathbb{P} = \{2,3,5,7,11,13,\dots\}$

This asymptotic result was conjenctured by the teenager C. F. Gauss in 1790's and first rigorously proved by J. Hadamard, and, independently, C. de la Vallée-Poussin in 1896.

We prove it by means of complex analysis, the Cauchy theorem ~~[illegible]~~, and will follow the proof of D. J. Newman in 1980. An important partial step was the theorem of P.L. Chebyshev in $\simeq 1850$ that $\pi(x) = \Theta\left(\frac{x}{\log x}\right)$, i.e. $\exists$ constants $0 < c_1 < c_2$ s.t. for every $x \geq 2$, $\frac{c_1 x}{\log x} < \pi(x) < \frac{c_2 x}{\log x}$

$\boxed{\text{Let's begin the proof (of PNT)}}$ $\boxed{\text{Proposition 1}}$ For $x \geq 2$,

$$\vartheta(x) := \sum_{p \leq x} \log p \leq (2\log 2)x.$$

$\boxed{\text{Proof.}}$ For $n \in \mathbb{N}$, ~~① $\binom{2n}{n} \geq \frac{4^n}{2n+1}$~~, ② $\binom{2n+1}{n} \leq 4^n$,

~~① [illegible], by the Cauchy theorem.~~

~~③ $\binom{2n}{n} \le (2n)^{\pi(2n)}$~~ and $\boxed{④\ \prod_{p\le n} p \le 4^n;}$ the claim clearly follows by setting $n=\lfloor x\rfloor$ and taking $\log$(④). ~~①~~ ~~and~~ ② follows from the binomial theorem: ~~$4^n=(1+1)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}$ and $\binom{2n}{k}\le\binom{2n}{n},\ 0\le k\le 2n$,~~

$$2\cdot 4^n=(1+1)^{2n+1}=\sum_{k=0}^{2n+1}\binom{2n+1}{k}\ge\binom{2n+1}{n}+\binom{2n+1}{n+1}=2\binom{2n+1}{n}.$$

~~③ This follows from the Exercise: $p^a \,\|\, \binom{2n}{n} \Rightarrow p^a\le 2n$ (p^a is the max. power of p dividing $\binom{2n}{n}$).~~ We prove ④ by induction on n. For $n=1,2$ it holds and so does for even $n>2$ as then $\prod_{p\le n} p=\prod_{p\le n-1} p$. Let $n=2m+1>1$ be odd. Then

$$\prod_{p\le n} p=\underbrace{\prod_{p\le m+1} p}_{\le 4^{m+1}\text{ by ind.}}\cdot\prod_{m+1<p\le 2m+1} p$$

where $\prod_{m+1<p\le 2m+1} p$ divides $\binom{2m+1}{m}$, hence is $\le 4^m$ by ②

$$\le 4^{m+1}\cdot 4^m=4^n$$

$\boxtimes$

~~We will use ① and ② later to prove Chebyshev's bounds.~~

$\boxed{\text{Proposition 2}}$ For $x\to+\infty$,

the PNT $\Longleftrightarrow \vartheta(x)=x+o(x)$,

i.e. $\pi(x)=\dfrac{x+o(x)}{\log x}\Longleftrightarrow\sum_{p\le x}\log p=x+o(x)$.

Proof. Indeed, $\frac{\vartheta(x)}{\log x} \le \pi(x) \le \frac{\vartheta(x)}{\log x}\left(1+O\left(\frac{\log\log x}{\log x}\right)\right) + \frac{x}{\log^2 x} =$

$= \frac{\vartheta(x)}{\log x} + O\left(\frac{x(\log\log x)}{(\log x)^2}\right)$. Here $\le$ follows from $\sum_{p\le x}\log p \le \pi(x)\log x$ and we get $\le$ from $\vartheta(x) \ge \sum_{y<p\le x}\log p \ge (\pi(x)-\pi(y))\log y$, so $\pi(x)\le \frac{\vartheta(x)}{\log y}+\pi(y) \le \frac{\vartheta(x)}{\log y}+y$, and we set $y=\frac{x}{\log^2 x}$. $\boxtimes$

Proposition 3 If the integral

$$\odot\int_1^{+\infty}\frac{\vartheta(x)-x}{x^2}\,dx = \int_0^{+\infty}\left(\vartheta(e^t)e^{-t}-1\right)dt$$

converges then $\vartheta(x)=x+o(x)$ for $x\to+\infty$ and (by Prop. 2) the PNT holds.

Proof. The 2nd $\int$ is obtained from the 1st one by the substitution $x=e^t$. Let $\vartheta(x)\ne x+o(x)$ for $x\to+\infty$. Thus, say, $\limsup_{x\to+\infty}\frac{\vartheta(x)}{x}>1$ (the case that $\liminf <1$ is similar). So $\exists\lambda>1\ \forall y>0\ \exists x:\ x>y \ \&\ \vartheta(x)>\lambda x$.

But then $\int_x^{\lambda x}\frac{\vartheta(t)-t}{t^2}\,dt > \int_x^{\lambda x}\frac{\lambda x - t}{t^2}\,dt = \int_1^{\lambda} \frac{\lambda-u}{u^2}\,du = c>0$, which means that the $\int\odot$ does not converge (the Cauchy cond. does not hold). $\boxtimes$

$\boxed{\text{Proposition 4}}$ For $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$,

$$\int_0^{+\infty} \left(\frac{\vartheta(e^t)}{e^t} - 1 \right) e^{-zt}\, dt = \frac{F(z+1)}{z+1} - \frac{1}{z}$$ where for $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, $F(s) := \sum_p \frac{\log p}{p^s}$.

$\boxed{\text{Proof.}}$ It suffices to show that when $\mathrm{Re}(s) > 1$,

$$s \int_0^{+\infty} \vartheta(e^t) e^{-st}\, dt = F(s),$$ because then we set $s := z+1$ and subtract $\int_0^{+\infty} e^{-zt} dt = \frac{1}{z}$.

Indeed, $$= s \int_1^{+\infty} \vartheta(x) x^{-s-1} dx = \sum_{n=1}^{\infty} \vartheta(n) \cdot s \int_n^{n+1} x^{-s-1} dx$$

$$= \sum_{n=1}^{\infty} \vartheta(n) \cdot (n^{-s} - (n+1)^{-s}) = \sum_{n=1}^{\infty} n^{-s} (\vartheta(n) - \vartheta(n-1)) =$$

$$= \sum_p \frac{\log p}{p^s} = F(s).$$ Exercise ~~Wh~~ Why? $\boxtimes$

The proof of the PNT rests on the next two results which we prove in the next (final) lecture.

$\boxed{\text{Proposition 5}}$ The function $\frac{F(z+1)}{z+1} - \frac{1}{z}$ $\left(= \frac{1}{z+1} \sum_p \frac{\log p}{p^{z+1}} - \frac{1}{z} \right)$ has a holomorphic extension from $\mathrm{Re}(z) > 0$ to $\mathrm{Re}(z) \geq 0$ (i.e. to some open set containing the closed half-plane $\mathrm{Re}(z) \geq 0$).

By its definition $F(s)$ is holom. on $Re(s) > 1$ as it is a Σ of entire functions and the Σ converges uniformly for $Re(s) > 1+\delta$, for any $\delta > 0$.

$\boxed{\text{Theorem 6}}$ (Wiener and (his student) Ikehara) in 1932

Let $f : [0, +\infty) \to \mathbb{R}$ satisfy: (i) f is bounded, (ii) $\int_a^b f$ exists for every $0 \le a < b < +\infty$, and (iii) $g(z) := \int_0^{+\infty} f(t) e^{-zt}\, dt$ (the Laplace transform of f) has a holom. extension from $Re(z) > 0$ to $Re(z) \geq 0$. Then $\int_0^{+\infty} f(t)\, dt$ converges and $= g(0)$ (i.e. we can set $z = 0$ in (iii)).

Function $g(z)$ is holom. on $Re(z) > 0$ by its definition.

$\boxed{\text{Proof of the PNT.}}$

We set $\underbrace{f(t) := \vartheta(e^t) e^{-t} - 1}^{\text{(in Thm. 6)}}$ and $g(z) := \frac{F(z+1)}{z+1} - \frac{1}{z}$.

Then f satisfies (i) by Prop. 1, (ii) by its definition and (iii) by Propositions 4 and 5. By Thm. 6,

$$\int_0^{+\infty} f = \int_0^{+\infty} \left(\frac{\vartheta(e^t)}{e^t} - 1 \right) dt$$ converges. By Propositions 3 and 2, the PNT follows. $\boxtimes$