$\boxed{\text{L 10, May 8, 2020}}$ | Properties of the ζ-function: extension to the meromorphic function $\zeta : \mathbb{C}\setminus\{1\} \to \mathbb{C}$ and the functional equation for ~~[illegible]~~. The function $\zeta(s)$ (zeta (s)) is defined in the half plane $Re(s)>1$ by the sum $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. Here $n^s = e^{s \log n}$ $\left(= \sum_{k=0}^{\infty} \frac{(s\log n)^k}{k!}\right)$, and since $|n^s| = n^{Re(s)}$, the series converges absolutely for $Re(s)>1$ and uniformly for $Re(s)>1+\delta$, for every $\delta>0$. By the standard theorem on holom. functions — a uniform sum of holom. functions is holom. — we get that

$$\zeta(s) : \{s\in\mathbb{C} \mid Re(s)>1\} \to \mathbb{C}$$

is a holomorphic function.

$\boxed{\text{Theorem}}$ (Thus we write $\zeta : \mathbb{C}\setminus\{1\} \to \mathbb{C}$.)

$\exists$ a meromorphic function $Z : \mathbb{C}\setminus\{1\} \to \mathbb{C}$ s.t.

(i) $Z(s) = \zeta(s)$ on the half-plane $Re(s)>1$.

(ii) $Res(Z(s), 1) = 1$. (iii) See below.

$\boxed{\text{Proof.}}$ I will follow in this passage on $\zeta(s)$ the excellent book "The theory of the Riemann Zeta-function" by E. C. Titchmarsh (in the edition revised by D. R. Heath-Brown).

(iii) $\boxed{\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s), \ \forall s \in \mathbb{C}\setminus\{1\}}$

Here $\Gamma(s)$ is the gamma function, one of which definitions is that for $Re(s)>0$, $s\in\mathbb{C}$, $\Gamma(s)=\int_0^{+\infty} e^{-x}x^{s-1}\,dx$.

(iii) is the famous functional equation for $\zeta(s)$ which relates values $\zeta(s)$ and $\zeta(1-s)$. Let us continue with the proof. We use theorem 2 from the 2^{nd} lecture:

If $a<b$ are real numbers and $f\in C^1(a,b)$, then

$$\sum_{a<n\le b} f(n)=\int_a^b f+\int_a^b (x-\lfloor x\rfloor-\tfrac{1}{2})f'(x)\,dx+(a-\lfloor a\rfloor-\tfrac{1}{2})f(a)-(b-\lfloor b\rfloor-\tfrac{1}{2})f(b).$$

I actually proved it without the "$-\frac{1}{2}$" but the proof (of the present form) is the same. (*) We apply this formula to the function $f(s)=n^{-s}$ with $s\in\mathbb{C}\setminus\{1\}$ and numbers $a,b\in\mathbb{N}$, $a<b$. So:

$$\sum_{n=a+1}^{b} f(n)=\frac{b^{1-s}-a^{1-s}}{1-s}-s\int_a^b\frac{x-\lfloor x\rfloor-\frac{1}{2}}{x^{s+1}}\,dx+\frac{b^{-s}-a^{-s}}{2}.$$

First we set $Re(s)>1$, $a=1$ and send $b\to+\infty$. Adding 1 to each side we get

(1)

$$\boxed{\zeta(s)=s\int_1^{\infty}\frac{\lfloor x\rfloor-x+\frac{1}{2}}{x^{s+1}}\,dx+\frac{1}{s-1}+\frac{1}{2}.}$$

this formula holds,

(*) Also, $f\in C^1(a,b)$ suggest that $f:[a,b]\to\mathbb{R}$ but the proof works also for $f:[a,b]\to\mathbb{C}$, which we need here.

as we assumed, for $Re(s)\geq 1$ but the right side in fact defines on the larger half-plane $Re(s)>0$ a merom. function

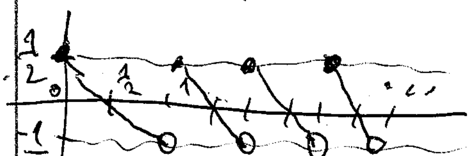
$Z:\{s\in\mathbb{C} \mid Re(s)>0\}\setminus\{1\}\to\mathbb{C}$ because $\lfloor x\rfloor - x+\frac{1}{2}$ is bounded and the $\int$ converges uniformly for $Re(s)>\delta>0$. (We use here a standard theorem on defining a holom. function by integration according to a parameter.)

Clearly, $Res(Z,1)=1$ and we proved (ii) and made a step to (i) (we extended $\zeta(s)$ to $Re(s)>0$, $s\neq 1$).

For $0<Re(s)<1$ we have that $s\int_0^1 \frac{\lfloor x\rfloor - x}{x^{s+1}}\,dx = -s\int_0^1 \frac{dx}{x^s} = \frac{s}{s-1}$, $\frac{s}{2}\int_1^\infty \frac{dx}{x^{s+1}} = \frac{1}{2}$ and therefore (1) can be written as

$$\boxed{\zeta(s)=s\int_0^\infty \frac{\lfloor x\rfloor - x}{x^{s+1}}\,dx \quad (0<Re(s)<1)}$$

But (1) extends $\zeta(s)$ even to $Re(s)>-1$: if $f(x)=\lfloor x\rfloor - x+\frac{1}{2}$, $f_1(x)=\int_1^x f(y)\,dy$, then $f_1(x)$ is bounded because $\forall q\in\mathbb{Z}$ we have $\int_q^{q+1} f(y)\,dy=0$ (f has graph



, here the "$\frac{1}{2}$" is explained).

So integrating by parts we obtain:

if $x_1 < x_2$ and $Re(s) > -1$ then $\int_{x_1}^{x_2} \frac{f(x)}{x^{s+1}} dx =$ (4)

$$= \left[\frac{f_1(x)}{x^{s+1}}\right]_{x_1}^{x_2} + (s+1)\int_{x_1}^{x_2} \frac{f_1(x)}{x^{s+2}} \to 0 \text{ as } x_1 \to +\infty.$$

Hence the $\int$ in (1) is convergent for $Re(s) > -1$.

It is easy to compute that $s\int_0^1 \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} dx =$

$= \frac{1}{s-1} + \frac{1}{2}$ $(Re(s) < 0)$. Thus from (1) we get

$$\boxed{\zeta(s) = s\int_0^\infty \frac{\lfloor x \rfloor - x + \frac{1}{2}}{x^{s+1}} dx \quad (-1 < Re(s) < 0).} \quad (2)$$

We have the Fourier series (3) $\lfloor x \rfloor - x + \frac{1}{2} = \sum_{n=1}^\infty \frac{\sin(2\pi n x)}{\pi n}$

We substitute it in (2) and integrate term by term:

$$\zeta(s) = \frac{s}{\pi} \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty \frac{\sin(2\pi n x)}{x^{s+1}} dx =$$

$$= \frac{s}{\pi} \sum_{n=1}^\infty \frac{(2\pi n)^s}{n} \cdot \int_0^\infty \frac{\sin y}{y^{s+1}} dy \quad (y := 2\pi n x)$$

$$= \frac{s}{\pi}(2\pi)^s \zeta(1-s) \cdot \sin\left(\frac{\pi s}{2}\right)\left(-\Gamma(\cancel{1}-s)\right).$$ Here

we used one of many identities for the gamma function, namely that $\int = -\sin\left(\frac{\pi s}{2}\right)\Gamma(-s)$. (4) If we recall the functional equation for $\Gamma(x)$, $x\cdot\Gamma(x)=\Gamma(x+1)$ (the factorial identity), we get that

$$\zeta(s)=2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)(-s)\Gamma(-s)\zeta(1-s)$$
$$=\boxed{2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)},$$

which is (iii). This holds if $-1<\mathrm{Re}(s)<0$, but the right side is in fact holom. for ~~...~~ $\mathrm{Re}(s)<0$. Thus we have holom. continuation of $\zeta(s)$ to the whole (punctured) plane $\mathbb{C}\setminus\{1\}$ and the functional equation (iii) holds there.

We still have to justify the above term by term integration.

Exercise We can integrate (3) term by term over any finite interval: $\int_a^b\left(\sum_{n=1}^{\infty}\cdots\right)dx=\sum_{n=1}^{\infty}\int_a^b(\cdots)\,dx$ for any real numbers $a<b$.

It therefore suffices to prove that

$$\lim_{\lambda\to\infty}\sum_{n=1}^{\infty}\frac{1}{n}\underbrace{\int_{\lambda}^{\infty}\frac{\sin(2\pi nx)\,dx}{x^{s+1}}}=0\quad(-1<\mathrm{Re}(s)<0).$$

Now $[\ldots]=\left[-\frac{\cos(2\pi nx)}{2\pi n\,x^{s+1}}\right]_{\lambda}^{\infty}-\frac{s+1}{2\pi n}\int_{\lambda}^{\infty}\frac{\cos(2\pi nx)}{x^{s+2}}dx$

(integration by parts) $=O\left(\frac{1}{n\,\lambda^{\sigma+1}}\right)+O\left(\frac{1}{n}\int_{\lambda}^{\infty}\frac{dx}{x^{\sigma+2}}\right)=$

$= O\left(\frac{1}{n x^{\sigma+1}}\right)$ where $\sigma = \mathrm{Re}(s)$. Thus the limit is 6

0. $\boxtimes$

I did not prove two results in the proof, namely the F. expansion (3) and the identity (4) for the gamma function: for $0 > \mathrm{Re}(s) > -1$ we have that $\int_0^\infty \frac{\sin x}{x^{s+1}}\,dx = -\Gamma(-s)\sin\left(\frac{\pi s}{2}\right)$.

The former is an exercise on Fourier series but unfortunately I do not have time to prove the latter.