Lecture 2. The result of Liouville and the Hermite–Hilbert theorem

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We say that $\alpha \in \mathbb{C}$ is an *algebraic* number if there is a nonzero polynomial $p \in \mathbb{Q}[x]$ such that $p(\alpha) = 0$. It is clear that we may take $p(x)$ to be monic (with the leading coefficient 1) or to be integral (in $\mathbb{Z}[x]$). If $\alpha \in \mathbb{C}$ is not algebraic, we say that it is a *transcendental* number. Today we prove the result of Liouville that for every $k \in \mathbb{N}$, $k \geq 2$, the number

$$\lambda(k) = \sum_{n=0}^{\infty} k^{-n!}$$

is transcendental, and we give the proof due to Hilbert of the theorem due to Hermite that the Euler number $\mathrm{e} = 2.71828\ldots$ is transcendental.

Liouville's result follows from the next so called *Liouville's inequality*.

Theorem 1 (J. Liouville, 1844) *For every algebraic number $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ there exist an $n \in \mathbb{N}$ and a constant $c > 0$ such that for every $\frac{p}{q} \in \mathbb{Q}$ it holds that*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{c}{q^n}\,.$$

Proof. We assume that $f \in \mathbb{Z}[x]$ is nonzero and has the minimum degree $n = \deg f(x)$ with respect to $f(\alpha) = 0$. Clearly, $n \geq 2$. We denote $I = [\alpha - 1, \alpha + 1]$ and take any fraction $\frac{p}{q}$; we may assume that $q \in \mathbb{N}$. The trivial case is when $\frac{p}{q} \notin I$. Then

$$\left|\alpha - \frac{p}{q}\right| \geq 1 \geq \frac{1}{q^n}\,.$$

The remaining case when $\frac{p}{q} \in I$ is more interesting. By the Lagrange mean value theorem there is a real number ζ that lies between α and $\frac{p}{q}$ and satisfies the equality

$$f(\alpha) - f(\tfrac{p}{q}) = f'(\zeta) \cdot (\alpha - \tfrac{p}{q})\,.$$

We denote $d = \max(\{|f'(x)| : \ x \in I\})$ (> 0), recall that $f(\alpha) = 0$ and get the bound

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{f(\frac{p}{q})}{d}\,.$$

We claim that $f(\frac{p}{q}) \neq 0$. If $f(\frac{p}{q}) = 0$ then for an appropriate number $N \in \mathbb{N}$ the product $g(x) = N \cdot \frac{f(x)}{x - p/q} \in \mathbb{Z}[x]$ would be an integral polynomial with $g(\alpha) = 0$ and $\deg g(x) = \deg f(x) - 1 = n - 1$, contradicting the definition of $f(x)$. Hence for $f(x) = \sum_{i=0}^{n} a_i x^i$ (where $a_i \in \mathbb{Z}$) we get the bound

$$|f(\tfrac{p}{q})| = q^{-n} \cdot \left| \textstyle\sum_{i=0}^{n} a_i p^i q^{n-i} \right| \geq q^{-n}$$

and $|\alpha - \frac{p}{q}| \geq \frac{1/d}{q^n}$. Combining the trivial bound and this bound, we get the desired bound that for every fraction $\frac{p}{q}$ it holds that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{\min(\{1,\, 1/d\})}{q^n}\,.$$

□

Corollary 2 (J. Liouville, 1844) *For every $k \in \mathbb{N}$, $k \geq 2$, the number*

$$\lambda(k) = \sum_{n=0}^{\infty} k^{-n!}$$

is transcendental.

Proof.

□

Transcendence of the Euler number e had been proven first by Hermite and then Hilbert simplified the proof. His proof rests on the property (E) of the Euler number that $(\mathrm{e}^x)' = \mathrm{e}^x$.

Theorem 3 (Ch. Hermite, 1873) *The Euler number $\mathrm{e} = 2.71\ldots$ is transcendental.*

Proof. (D. Hilbert, 1890) It is not hard to compute using (E) and integration by parts that for every $n \in \mathbb{N}_0$,

$$\int_0^{+\infty} x^n \cdot \mathrm{e}^{-x}\, dx = n!\,.$$

We have more generally for any integral polynomial $p(x) = \sum_{i=0}^{n} a_i x^i$, where $a_i \in \mathbb{Z}$ and $n \in \mathbb{N}_0$, that $I = \int_0^{+\infty} p(x) \cdot \mathrm{e}^{-x}\ dx = \sum_{i=0}^{n} a_i \cdot i! \in \mathbb{Z}$. If $a_0 = a_1 = \cdots = a_k = 0$ then $I \equiv 0 \pmod{(k+1)!}$.

Suppose for the contrary that e is algebraic and there is a nonzero integral polynomial $p \in \mathbb{Z}[x]$ such that $p(\mathrm{e}) = 0$. It follows that then there exist integers $a_0, a_1, \ldots, a_n$ with $n \in \mathbb{N}_0$ and $a_0 \neq 0$ such that

$$\sum_{i=0}^{n} a_i \mathrm{e}^i = 0\,.$$

□

References

[1] V. M. Schmidt, *Diophantine Approximation*, Springer-Verlag, Berlin 1980

[2] V. Šmidt, *Diofantovy priblizheniya*, Mir, Moskva 1983