MATHEMATICAL STRUCTURES (NMAI064)
summer term 2021/22
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**LECTURE 13 (5/16/2022).** TOPOLOGY: CONNECTED AND DISCONNECTED SPACES
(based on the lecture notes of A. Pultr, Chapter V.7)

• *Connected and disconnected spaces.* A set $A \subseteq X$ in a topological space $(X, \tau)$ is *clopen* if it is closed and open, i.e. both $A \in \tau$ and $X \backslash A \in \tau$. Clearly, both $\emptyset$ and $X$ are clopen. A space $X$ is *connected* if $X \neq \emptyset$ and the only clopen sets in it are just those two, $\emptyset$ and $X$. Else, if there is a nontrivial clopen set $A \subseteq X$, $A \neq \emptyset, X$, in $X$, we say that $X$ is *disconnected*. (So we exclude $X = \emptyset$ from our considerations. We omit checks of this case in the forthcoming proofs and leave them to the reader as exercises.) A subset $Y \subseteq X$ is *connected* if the subspace induced on $Y$ is connected.

**Exercise 1** *Let $Y \subseteq X$ and $(X, \tau)$ be a topology. Show that the subset $Y$ is disconnected if and only if*

$$\exists\, A,\, B \in \tau:\ (A \cap Y,\, B \cap Y \neq \emptyset) \wedge (Y \subseteq A \cup B) \wedge$$
$$\wedge\, (A \cap Y) \cap (B \cap Y) = \emptyset\ .$$

*We say that $A$* and *$B$* rip *$Y$. The same equivalence holds with closed sets $A$ and $B$.*

**Exercise 2** *Prove that a subset $Y \subseteq \mathbb{R}$ is connected (in the Euclidean topology) if and only if $Y$ is an interval, i.e. $Y$ has the property that*

$$(x,\, y,\, z \in \mathbb{R}) \wedge (x < y < z) \wedge (x, z \in Y) \Rightarrow y \in Y\ .$$

**Theorem 3 (continuous images)** *All continuous images of connected sets are connected.*

**Proof.** Suppose that $(X, \tau)$ and $(Z, \sigma)$ are topological spaces, that $Y \subseteq X$ and that $f\colon X \to Z$ is continuous. We show that if $f[Y]$ is disconnected in $Z$ then $Y$ is disconnected in $X$. Thus we assume that we have sets $A, B \in \sigma$ ripping $f[Y]$ (as described in Exercise 1). It is easy to check (we are lazy to write it down) that the sets

$$f^{-1}[A],\ f^{-1}[B] \in \tau$$

rip $Y$ and so $Y$ is disconnected. $\square$

**Theorem 4 (on closure)** *Closure of a connected set is connected.*

**Proof.** Suppose that $(X, \tau)$ is a topological space, $Y \subseteq X$ and that $\overline{Y}$ is disconnected. We take closed sets $A$ and $B$ in $X$ ripping $\overline{Y}$ and claim that they also rip $Y$ and so that $Y$ is disconnected. The required properties of $A$ and $B$ with respect to $Y$ (see Exercise 1) are all obviously satisfied from the relation that $Y \subseteq \overline{Y}$, except maybe for the property that both $A$ and $B$ should intersect $Y$. Suppose for the contrary that, say, $Y \subseteq A$ and $Y \cap B = \emptyset$. But then since $A$ is closed, also $\overline{Y} \subseteq A$ and $\overline{Y} \cap B = \emptyset$, contradicting the assumption that $A$ and $B$ rip $\overline{Y}$. $\square$

• *Connectedness and products.* We show that any product of connected spaces is connected. For the proof we need a basically combinatorial result.

**Theorem 5 (combinatorial)** *Suppose that $(X, \tau)$ is a topology and that $X_i \subseteq X$, $i \in J$, are connected subsets such that for every $i, j \in J$ there exist finitely many indices $i_0, i_1, \ldots, i_n \in J$, $n \in \mathbb{N}$ such that*

$$i_0 = i \wedge i_n = j \wedge (\forall\, k = 0,\, 1,\, \ldots,\, n-1:\ X_{i_k} \cap X_{i_{k+1}} \neq \emptyset)\ .$$

*Then the set $Y := \bigcup_{i \in J} X_i \subseteq X$ is connected.*

**Proof.** As usual, we prove the reversal; we assume that the union $Y$ is disconnected and deduce a contradiction. Suppose that $A, B \in \tau$ rip the union $Y$. It follows that for every $i \in J$, either $X_i \subseteq A$ and $X_i \cap B = \emptyset$, or $X_i \subseteq B$ and $X_i \cap A = \emptyset$. Let $J_1 \subseteq J$, resp. $J_2 \subseteq J$, be the set of indices $i \in J$ for which the former, resp. the latter, case occurs. The sets $J_1$ and $J_2$ partition $J$ and are nonempty. We take a $j_1 \in J_1$ and a $j_2 \in J_2$ and apply to these two indices the hypothesis on the set system $X_i$, $i \in J$. It follows that there exist indices $j_3 \in J_1$ and $j_4 \in J_2$ such that $X_{j_3} \cap X_{j_4} \neq \emptyset$. But since $X_{j_3} \subseteq A$ and $X_{j_4} \subseteq B$, we see that the sets $A \cap Y$ and $B \cap Y$ intersect, contradicting the assumption that $A$ and $B$ rip $Y$. $\square$

**Theorem 6 (on products)** *The product* $X := \prod_{i \in J}(X_i, \tau_i)$ *of any system of connected topological spaces is connected.*

**Proof.** We begin with the case $J = \{1, 2\}$. We fix a point $x \in X_1$ and consider the subsets

$$\{x\} \times X_2 \text{ and } X_1 \times \{y\},\ y \in X_2\ ,$$

in $X_1 \times X_2$. Their union equals $X_1 \times X_2$, the former set intersects each of the latter sets, the former set is homeomorphic to $X_2$ and hence connected, and similarly for each of the latter sets. By the previous theorem, the product $X_1 \times X_2$ is connected. Thus also any finite product of connected spaces is connected.

We consider the case with a general index set $J$. For every $i \in J$ we fix a point $a_i \in X_i$, and for every finite set of indices $K \subseteq J$ set

$$X_K := \{(x_i)_{i \in J} \mid i \in J \setminus K \Rightarrow x_i = a_i\} \subseteq X\ .$$

The (sub)space $X_K$ is homeomorphic with $\prod_{i \in K} X_i$ (Exercise 7) and hence connected. Clearly, for any finite sets $K, K' \subseteq J$ we have that $X_K, X_{K'} \subseteq X_{K \cup K'}$. Thus by Theorem 5, the set

$$M := \bigcup_{\substack{K \subseteq J \\ K \text{ finite}}} X_K \subseteq X$$

is connected. The set $M$ is dense in $X$ because any nonempty basis set $\bigcap_{i \in K} p_i^{-1}[U_i]$, where $K \subseteq J$ is finite and $U_i \in \tau_i$, intersects the set $X_K \subseteq M$. Thus, by Theorem 4, the set $\overline{M} = X$ is connected. $\square$

**Exercise 7** *Prove that the space* $X_K$ *in the previous proof is homeomorphic with the corresponding finite product* $\prod_{i \in K} X_i$.

**Exercise 8** *Prove that if the product* $\prod_{i \in J}(X_i, \tau_i)$ *is connected, then every space* $(X_i, \tau_i)$ *is connected.*

• *Path-wise and arc-wise connected spaces.* We say that a topological space $(X, \tau)$ is *path-wise connected* if

$$\forall\, x,\ y \in X\ \exists \text{ a continuous map } f\colon [0,\ 1] \to X :\ f(0) = x \wedge f(1) = y\ .$$

Here the real interval $[0, 1]$ has the Euclidean topology and $f$ need not be injective. We say that $(X, \tau)$ is *arc-wise connected* if in the above, the map $f$ is additionally required to be injective and only points $x \neq y$ are considered. These maps $f$ (or their images) are called *paths* and *arcs* (joining the points $x$ and $y$), respectively.

**Exercise 9** *Show that every arc-wise connected space is path-wise connected.*

**Exercise 10** *Show that every arc $f\colon [0,1] \to X$ is in fact a homeomorphism from $[0,1]$ to the subspace $f[\,[0,1]\,] \subseteq X$.*

In graph theory, we call a *walk* what is a path in topology, and — in the injective case — a *path* what is an arc in topology. The terminology in graph theory is not completely standardized, though.

**Exercise 11** *Recall the argument that in graph theory both kinds of connectedness coincide: a graph $G = (V, E)$ is connected iff every two vertices in $V$ can be joined by a walk iff every two vertices in $V$ can be joined by a path.*

It is different in topology: in the article *Connected space* in Wikipedia, `https://en.wikipedia.org/wiki/Connected_space`, one can find a simple example of a non-Hausdorff space that is path-wise connected but is not arc-wise connected, and also the following claim.

**Theorem 12 (paths versus arcs)** *Every Hausdorff space that is path-wise connected is also arc-wise connected.*

The proof — I found some remarks and hints concerning it on the Internet but nothing really clear — seems to be nontrivial.

**Proposition 13 (p.-w. conn. $\Rightarrow$ conn.)** *Any path-wise connected space $(X, \tau)$ is connected.*

**Proof.** We suppose that $(X, \tau)$ is path-wise connected, fix a point $x \in X$, and consider the sets

$$f_y[\,[0,\,1]\,] \subseteq X,\ y \in X\ ,$$

where each $f_y\colon [0,1] \to X$ is a path joining $x$ and $y$. These sets share the point $x = f_y(0)$, each is connected by Theorem 3 and Exercise 2, and their union is the whole $X$ (since $f_y(1) = y$). Hence by Theorem 5, the space $X$ is connected. $\square$

**Theorem 14 (on open sets in $\mathbb{R}^n$)** *Every connected open set $U \subseteq \mathbb{R}_n$ (in the Euclidean topology) is arc-wise connected.*

**Proof.** Consider the relation $\sim$ on a connected open set $U \subseteq \mathbb{R}^n$, defined for $x, y \in U$ with $x \neq y$ by

$$x \sim y \iff \exists \text{ an arc in } U \text{ joining } x \text{ and } y \ ,$$

and enlarged by all diagonal pairs $x \sim x$, $x \in U$. It is not hard to show (in Exercise 15) that $\sim$ is an equivalence relation and that every block (of $\sim$)

$$[x] = \{y \in U \mid x \sim y\}, \ x \in U \ ,$$

is an open set. If there were at least two blocks, we would have for $U$ a partition

$$U = V \cup W$$

in two disjoint nonempty open sets: $V$ is an arbitrary block and $W$ is the union of the other blocks. But then $U$ would be disconnected, which is not the case. So there exists only one block of $\sim$ and $U$ is arc-wise connected. $\square$

**Exercise 15** *Prove that the relation $\sim$ in the previous proof is an equivalence relation and has open blocks.*

**Exercise 16** *Prove that every connected set $Y \subseteq \mathbb{R}$ (in the Euclidean topology) is arc-wise connected.*

In two dimensions the situation is different.

**Theorem 17 (a peculiar closed plane set)** *The closed set*

$$P = A \cup B := \big(\{0\} \times [-1,\, 1]\big) \cup \{(t,\, \sin(1/t)) \mid 0 < t \leq 1\} \subseteq \mathbb{R}^2$$

*is connected but is not path-wise connected.*

**Proof.** Since $P = \overline{B}$ and $B$ is connected (it is a continuous image of an interval), by Theorem 4 the set $P$ is connected.

We show that no path in $P$ joins any $a \in A$ and $b := (1, \sin 1) \in B$. Let $a \in A$ be arbitrary and suppose for the contrary that

$$f = f(t) = (f_x(t),\, f_y(t))\colon [0,\, 1] \to P$$

is a continuous map such that $f(0) = a$ and $f(1) = b$. We may assume that $f(t) \in B$ for every $t \in (0, 1]$; else we would replace $[0, 1]$ with the interval $[\alpha, 1]$, where $\alpha$ is the supremum of $t \in [0, 1]$ such that $f(t) \in A$. We claim that there are two sequences

$$1 \geq u_1 > u_2 > \cdots > 0 \quad \text{and} \quad 1 \geq v_1 > v_2 > \cdots > 0$$

that both go in limit to 0 and are such that for every $i \in \mathbb{N}$,

$$f_y(u_i) = -1 \quad \text{and} \quad f_y(v_i) = 1 \; .$$

The claim on the $u_i$s follows from the fact that for every $u \in (0, 1]$ there exists a $u' \in (0, u)$ such that $f_y(u') = -1$. Indeed, we take a large enough $k \in \mathbb{N}$ such that

$$t_1 := \frac{1}{(2k - 1/2)\pi} < f_x(u) \; .$$

Since $\lim_{t \to 0} f_x(t) = 0$, we can find a $t_2 \in (0, u)$ such that $f_x(t_2) < t_1$. The function $f_x \colon [t_2, u] \to \mathbb{R}$ is continuous, and hence attains at some $u' \in (t_2, u)$ the intermediate value

$$f_x(u') = t_1 \; .$$

But then

$$f_y(u') = \sin(1/t_1) = -1 \; ,$$

as claimed. The claim on the $v_i$s follows by a similar argument. Thus the limit

$$\lim_{t \to 0} f_y(t)$$

does not exist, which contradicts the fact that it should be equal to $a_y \in [-1, 1]$, where $a = (0, a_y)$. □

• *Around Jordan's theorem.* There are several famous and difficult-to-prove results on connected and disconnected sets in the Euclidean plane $\mathbb{R}^2$. We mention two, without proofs (maybe we will say more on them in the next year version of this course). After that we give, with a proof, a shocking example.

Let $(X, \tau)$ be a topological space. A *(connected) component* of $X$ is any inclusion-wise maximal connected subset $Y \subseteq X$.

**Exercise 18** *Prove that components partition the space (i.e., they are disjoint and nonempty and their union is the whole space).*

A *(topological) circuit* (or a *loop*) in $X$ is any "closed arc" in $X$, i.e., any continuous map

$$f\colon [0,\, 1] \to X$$

such that $f(0) = f(1)$ but the restriction $f \mid (0,1]$ is injective.

**Theorem 19 (C. Jordan, 1887)** *For every circuit* $f\colon [0,1] \to \mathbb{R}^2$*, the subspace*

$$\mathbb{R}^2 \setminus f[\,[0,\, 1]\,]$$

*of the Euclidean plane* $\mathbb{R}^2$ *has exactly two components. They are both open, one of them is bounded and is called the* interior (of $f$)*, and the other is unbounded and is called the* exterior (of $f$)*.*

**Exercise 20** *Prove that (i) every component of the complement of a loop in* $\mathbb{R}^2$ *is open and (ii) exactly one of the components is unbounded.*

*Camile Jordan (1838–1922)* proved his theorem in his textbook of mathematical analysis *Cours d'analyse de l'École Polytechnique*. By the interesting Wikipedia article *Jordan curve theorem*, `https://en.wikipedia.org/wiki/Jordan_curve_theorem`, "For decades, mathematicians generally thought that this proof was flawed and that the first rigorous proof was carried out by *Oswald Veblen (1880–1960)*. However, this notion has been overturned by *Thomas C. Hales (1958)* and others."

Many textbooks of discrete mathematics prove *Euler's formula*

$$|V| - |E| + |F| = 2\ ,$$

which relates the numbers of vertices $V$, edges $E$, and faces $F$ in a drawing of any *connected* plane graph, and mention that the proof relies on Jordan's theorem. It seems that not so many people realize that besides Jordan's theorem, another fundamental result on arcs in the plane is needed for Euler's formula, namely the following.

**Theorem 21 (the arc theorem)** *For every arc $f\colon [0,1] \to \mathbb{R}^2$, the subspace*

$$\mathbb{R}^2 \setminus f[\,[0,\,1]\,]$$

*of the Euclidean plane $\mathbb{R}^2$ is connected.*

Without this theorem you would not be able to show that the one-edge plane graph has only one face.

**Theorem 22 (a shocking example)** *Let*

$$K := [0,1] \times [0,1] \subseteq \mathbb{R}^2$$

*be the Euclidean unit square and let $A := \{0\} \times [0,1]$, $B := [0,1] \times \{0\}$, $C := \{1\} \times [0,1]$ and $D := [0,1] \times \{1\}$ be its left, bottom, right and top side, respectively. There exists a partition*

$$K = X \cup Y$$

*into connected sets $X$ and $Y$ such that $X \cap Y = \emptyset$ and*

$$X \cap A \neq \emptyset \;\wedge\; X \cap C \neq \emptyset \;\wedge\; Y \cap B \neq \emptyset \;\wedge\; Y \cap D \neq \emptyset\,.$$

**Proof.** We consider the partition

$$[0,\,1] = I \cup J := ([0,\,1] \cap \mathbb{Q}) \cup \big([0,\,1] \setminus \mathbb{Q}\big)$$

and set

$$X := \big(I \times [0,\,1)\big) \cup \big(J \times \{1\}\big) \;\text{ and }\; Y := K \setminus X.$$

It is clear that $X$ and $Y$ partition $K$, that $X$ intersects both $A$ and $C$, that $Y$ intersects both $B$ and $D$, and that both $X$ and $Y$ are connected (Exercise 23). □

**Exercise 23** *Prove that the sets $X$ and $Y$ in the previous proof are connected.*

Thus it is possible to go in $K$, in a connected way, first from $A$ to $C$ and then from $B$ to $D$ in such a way that the two journeys do not intersect! However, we have of course the following result on journeys in $K$ that use arcs.

**Exercise 24** *Suppose that*

$$f,\, g\colon [0,\, 1] \to K$$

*are two arcs such that* $f(0) \in A$*,* $f(1) \in C$*,* $g(0) \in B$ *and* $g(1) \in D$*, but these (at most) four points are the only intersections of the images of* $f$ *and* $g$ *with the boundary of the square* $K$*. Prove that then these images intersect,*

$$f[\,[0,\, 1]\,] \cap g[\,[0,\, 1]\,] \neq \emptyset\ .$$

*Hint: use Theorems 19 and 21 (maybe not even these suffice for the proof).*

• *Locally connected spaces, totally and extremally disconnected spaces.* A space $(X, \tau)$ is *locally connected* if for every point $x \in X$ and every open set $U \ni x$ there exist sets $V \in \tau$ and $K \subseteq X$ such that $K$ is connected and

$$x \in V \subseteq K \subseteq U\ .$$

In words, every neighborhood of every point contains its connected neighborhood.

**Exercise 25** *Show that the space* $P$ *in Theorem 17 (which is connected) is not locally connected.*

A space $(X, \tau)$ is *totally disconnected* if every of its components is a point. A space $(X, \tau)$ is *extremally disconnected* if the closure of every open set in $X$ is open and hence clopen).

**Exercise 26** *Prove that the Euclidean subspace* $\mathbb{Q} \subseteq \mathbb{R}$ *is totally disconnected.*

THANK YOU!

To get "zápočet" for the tutorial, you should solve (or at least send in solutions of) at least half of the homework exercises.

Here are six questions for the exam. I will examine only these, i.e. not the first part of the course taught by A. Pultr.

1. Show that AC implies the existence of a non-measurable set — Cor. 11 in l. 9.

2. Explain the prophet paradox — Thm. 21 in l. 9.
3. Explain ways to introduce topology on a set and prove that the Sorgenfrey line has no countable basis — l. 11 and Prop. 13 in l. 10.
4. Explain the hierarchy of separation axioms for topological spaces and prove that metrizable topology is normal — l. 11 and Ex. 19 in l. 11.
5. Define compact spaces, prove Tikhonov's theorem and give some application of it — l. 13 and Thms. 8 and 10 in l. 12.
6. Define connected and path-wise connected spaces and give an example of a space that is connected but not path-wise connected — l. 14 and Thm. 17 in l. 13.