Notation and conventions. Standard, but a mapping

$$f : X \to Y$$

is not just a subset of $X \times Y$. We include the information on X and Y (hence it makes sense to distinguish the mappings that are *onto* (that is, every $y \in Y$ appears in $(x, y) \in f$. We write $f(x)$ in the obvious sense. It is useful to think of f as a symbol for a formula (which we may or may not have) for associating values in Y with arguments in X.

If X resp. Y are endowed with a structure then the structures are included in the information (thus for instance it makes sense to ask whether f is continuous in case of spatial structures).

The X is referred to as the *domain*, the Y as the *range*.

If one has a formula F for the mapping we often write $f = (x \mapsto F(X))$ like for instance in $f = (x \mapsto x^2) : \mathbb{R} \to \mathbb{R}$.

If $f : X \to Y$ and $g : Y \to Z$ we have the *composition*

$$g \circ f = (x \mapsto g(f(x))) : X \to Z.$$

We often write just $g \cdot f$ or gf for $g \circ f$.

If $f : X \to Y$ is one-one and onto we have the *inverse*

$f^{-1} : Y \to X$ determined by

$$ff^{-1} = \mathrm{id}_Y \text{ and } f^{-1}f = \mathrm{id}_X$$

(The id's are the identity mappings $(x \mapsto x)$.)

For $f : X \to Y$, $A \subseteq X$ and $B \subseteq Y$ we have

the image $f[A] = \{f(x) \mid x \in A\}$ and
the preimage $f^{-1}[B] = \{x \mid f(x) \in B\}$.

Note that obviously

$$f[f^{-1}[B]] \subseteq B \quad \text{and} \quad f^{-1}[f[A]] \supseteq A.$$

To observe.

1. When one has $\forall B, f[f^{-1}[B]] = B$?
2. When one has $\forall B, f^{-1}[f[A]] = A$?
3. $f^{-1}[B]$ is defined for any f. If there exists the inverse map, the symbol can be read in two ways. Can it create confusion?

Binary relation $R \subseteq X \times X$, and homomorphisms $f : (X, R) \to (Y, S)$ satisfying

$$(x_1, x_2) \in R \;\Rightarrow\; (f(x_1), f(x_2)) \in S.$$

More generally,
Unary, ternary, n-ary relations $R \subseteq X$, $R \subseteq X \times X \times X$,

$$R \subseteq \overbrace{X \times \cdots \times X}^{n \text{ times}},$$

homomorphisms following the rule

$$(x_1, \ldots, x_n) \in R \;\Rightarrow\; (f(x_1), \ldots, f(x_n)) \in S.$$

and infinitary ones.
The notation starts to be complicated.
More expedient: use the convention:

$$X^A = \{\xi \mid \xi : A \to X\}$$

Then e.g. $X \times X$ is represented as $X^{\{1,2\}}$, (x_1, x_2) is the symbol (in fact, table) of the map $(i \mapsto x_i)$.

Thus we have *A-nary relations*

$$R \subseteq X^A$$

and *homomorphisms*

$$f : (X, R) \to (Y, S)$$

satisfying the formula

$$\xi \in R \;\Rightarrow\; f \circ \alpha \in S.$$

Check that for binary, ternary, n-ary relations this agrees with the previous and realize how much easier it is to work with!

Just a triviality: the proof of the fact that the composition of homomorphisms is a homomorphism is now expressed by the associativity

$$(g \circ f) \circ \xi = g \circ (f \circ \xi).$$

Subobject. Consider an (X, R) with an A-nary relation $R \subseteq X^A$ and a subset $Y \subseteq X$. Denote by $j : Y \to X$ the embedding map $j = (x \mapsto x)$. Set

$$R_Y = \{\beta : A \to Y \mid j\beta \in R\}$$

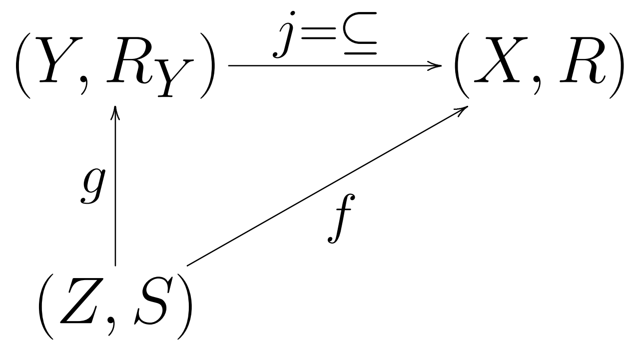
Realize that

(1) R_Y is in a natural one-one correspondence with $\{\alpha \in R \mid \alpha[A] \subseteq Y\}$

(2) R_Y is the largest A-nary relation on Y such that $j : Y \subseteq X$ is a homomorphism.

(3) Represent the edges in a graph as a binary relation R on the set of vertices X. Then (Y, R_Y) represents the induced subgraph on the set of vertices Y.

Proposition. *In the following diagram*



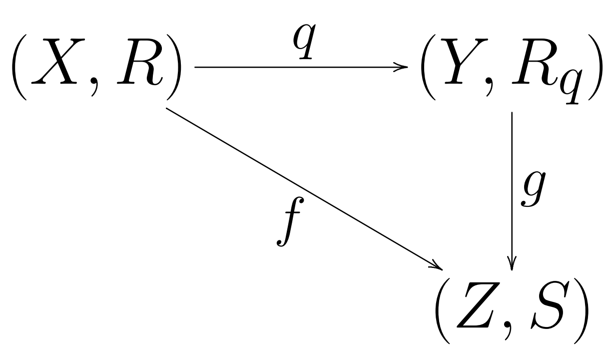
let f be a homomorphism and let $jg = f$. Then g is a homomorphism.
Proof is straightforward: if α is in S then $f\alpha = (jg)\alpha = j(g\alpha)$ is in R and hence $g\alpha$ is in R_Y. $\square$

The fact that j is an actual embedding is inessential. We can consider any one-one mapping $j : Y \to X$ and define $R_j = \{\beta : A \to Y \mid j\beta \in R\}$. One usually speaks on subobjects in thus generalized situation.

Quotients, factorobjects. Dualy, for (X, R) and an onto map $q : X \to Y$ one defines $R_q = \{q\alpha \,|\, \alpha \in R\}$ on Y and obtains the *smallest* A-nary relation on Y such that q is a homomorphism.

One has

Proposition. *In the following diagram*



let f *be a homomorphism and let* $gq = f$*. Then* g *is a homomorphism.*

We speak of such (Y, R_q) as of *quotients* or *factorobjects* of (X, R).

Products. Let R_i be A-nary relations on X_i, $i \in J$. On the cartesian product $X = \prod_J X_i$ with projections $p_j : \prod X_i \to X_j$ define an A-nary relation

$$R = \{\alpha : A \to X \mid \forall i, p_i\alpha \in R_i\}$$

$(\prod X_i, R)$ is called the *product* of the system (X_i, R_i), $i \in J$ and denoted by

$$\prod_{i \in J} (X_i, R_i).$$

Note that

R is the largest relation on the cartesian product $\prod_J X_i$ such that all the projections

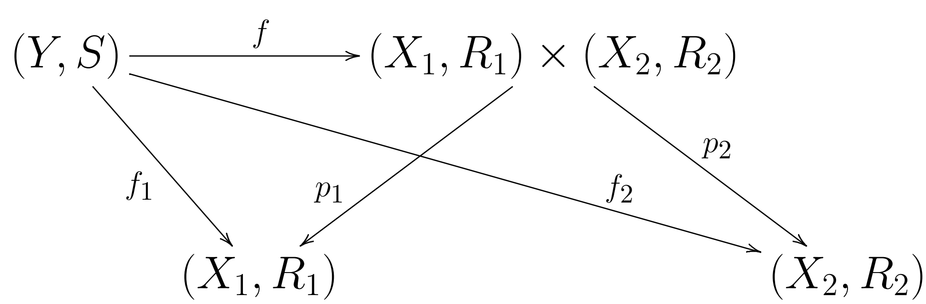
$$p_j : (\prod_J X_i, R) \to (X_j, R_j)$$

are homomorphisms.

Proposition. *For any system of homomorphisms $f_i : (Y, S) \to (X_i, R_i)$ there is a unique homomorphism $f : (Y, S) \to \prod_J (X_i, R_i)$ such that $\forall i$, $p_i f = f_i$.*

Proof. Define $f : Y \to \prod_i X_i$ by $f(y) = (f_i(y))_i$. Obviously this is the unique *mapping* such that $p_i f = f_i$. It is a homomorphism: if $\alpha : A \to Y$ is in S, each $f_i\alpha$, that is $(p_i f)\alpha = p_i(f\alpha)$, is in R_i and hence $f\alpha \in R$.

Let us visualize the situation for a product of two:



Relational systems and objects.
A *type* is a system

$$\Delta = (A_t)_{t \in T}$$

A *relational system* of the type Δ on a set X is a system

$$R = (R_t)_{t \in T} \quad \text{of } R_t \ A_t\text{-nary relations on } X \quad .$$

Of the pair (X, R) we then speak of as of a relational object (of the type Δ).

Everything we have introduced for individual relations is extended to relational objects coordinatewise (e.g. a subobject on Y of (X, R) is endowed with $R_Y = ((R_t)_Y)_t$, etc. and we have the propositions extended in the obvious way.

Preorder and order

A *preorder* on X: a relation $R \subseteq X \times X$ that is

- reflexive, that is, xRx for all $x \in X$,
- and transitive, that is, xRy and yRz implies xRz.

If xRy and $yRx \quad \Rightarrow \quad x = y$, we speak of a *(partial) order* and of $(X, \leq)$ as of a (p.) ordered set, briefly *poset*.

If for all x, y either xRy or yRx we speak of a *linear order* or a *chain*.

Out of a preordered set (X, R) we easily obtain an ordered one by introducing the equivalence

$$x \sim y \quad \equiv \quad xRy \;\&\; yRx.$$

An unspecified order is usually denoted by $\leq$. Other symbols according to the situation e.g. $\leq_1$, $\leq'$, $\preceq$, $\sqsubseteq$ etc..

Further notation

$$\downarrow x = \{y \mid y \leq x\},\ \uparrow x = \{y \mid y \geq x\},$$

$$\downarrow M = \bigcup_{x \in M} \downarrow x,\ \uparrow M = \bigcup_{x \in M} \uparrow x.$$

Examples abound, e.g.

(a) standard linear orderings of numbers

(b) divisibility of integers ($a|b$, "a divides b") is a preorder,

(c) the inclusion is a (partial) order on the set $\mathfrak{P}(X)$ of all subsets of a set X.

Opposite (dual) order:

$a \leq^{\mathrm{op}} b$ iff $b \leq a$

We often write

$$(X, R)^{\mathrm{op}} \quad \text{for} \quad (X, R^{\mathrm{op}}).$$

Monotone maps. If $(X, \leq)$, $(Y, \leq)$ are posets (the two $\leq$, of course, do not have to coincide) and if $f : X \to Y$ is a mapping, we say that f is *monotone* (or *isotone*) if

$$x \leq y \quad \Rightarrow \quad f(x) \leq f(y).$$

An monotone f is an *isomorphism*, if there exits an monotone $g : (Y, \leq) \to (X, \leq)$ such that $f \cdot g = \mathrm{id}$ a $g \cdot f = \mathrm{id}$, We immediately see that f is an isomorphism iff

- it is a mapping onto, and
- $x \leq y \quad \Leftrightarrow \quad f(x) \leq f(y).$

Suprema and infima

$x \in (X, \leq)$ is a *lower* (resp. *upper*) *bound* of $M \subseteq X$ if

$$M \subseteq\uparrow x \quad (\text{resp. } M \subseteq\downarrow x).$$

The least upper bound of M (if it exists) is called *supremum* of M, denoted

$$\sup M,$$

the largest lower bound of M is called *infimum* and denoted

$$\inf M.$$

Thus, $s = \sup M$ if

(1) $M \subseteq\downarrow s$, and

(2) $M \subseteq\downarrow x \;\Rightarrow\; s \leq x$.

Compare with the

(2') $\quad x < s \quad \Rightarrow \quad \exists y \in M, x < y.$

from analysis courses.

Let $(X, \leq)$ be a poset and let $M \subseteq N \subseteq X$. We say that M is *up-* resp. *down-cofinal* in N if for each element $n \in N$ there is an $m \in M$ such that $m \geq n$ resp. $m \leq n$. One often uses the following

Observation. *If M is up- (resp. down-) cofinal with N and* $\sup N$ *(resp.* $\inf N$*) exists then* $\sup M$ *(resp.* $\inf M$*) exists as well and* $\sup M = \sup N$ *(resp.* $\inf M = \inf N$*).*

Proposition. *We have*

$$\sup\{\sup M_j \mid j \in J\} = \sup(\bigcup_{j\in J} M_j),$$

$$\inf\{\inf M_j \mid j \in J\} = \inf(\bigcup_{j\in J} M_j)$$

whenever the left hand sides make sense.

Proof for suprema. Set

$$s_j = \sup M_j, \quad s = \sup\{s_j \mid j \in J\}.$$

Then s is obviously an upper bound of the set $\bigcup_{j\in J} M_j$. Now if $\bigcup_{j\in J} M_j \subseteq {\downarrow}x$ we have for each j, $M_j \subseteq {\downarrow}x$ and hence $s_j \leq x$. Consequently $\{s_j \mid j \in J\} \subseteq {\downarrow}x$ and finally $s \leq x$.

Bottom and top. $\sup\emptyset$ (if it exists) is the least element (notation: $\perp$, 0) of $(X \leq)$ ($\emptyset \subseteq\downarrow x$ for every x). Similarly $\inf\emptyset$ is the largest element ($\top$,1)

Note the least element is minimal, but a *minimal* element (such that implication $y \leq x \Rightarrow y = x$) is not necessarily least. Similarly for maximal and largest elements.

Examples. (a) Suprema and infima in $\mathbb{R}$ as in analysis.

(b) In $(\mathfrak{P}(X), \subseteq)$ we have

$$\sup\{A_j \mid j \in J\} = \bigcup_{j\in J} A_j, \quad \inf\{A_j \mid j \in J\} = \bigcap_{j\in J} A_j.$$

(c) In $\mathbb{N}$ with $a|b$ ("a divides b"), $\sup\{a, b\}$ is the least common multiple of a and b and $\inf\{a, b\}$ is the largest common divisor of a and b.

If $f : (X \leq) \to (Y, \leq)$ is monotone then

$$f[{\downarrow}x] \subseteq {\downarrow}f(x) \quad \text{and} \quad f[{\uparrow}x] \subseteq {\uparrow}f(x),$$

so that if x is an upper (lower) bound of M then $f(x)$ is an upper (lower) bound of $f[M]$. Hence in particular

$$\sup f[M] \leq f(\sup M), \quad \inf f[M] \geq f(\inf M)$$

(whenever the expressions make sense).

Some special orders

Semilattices. A *lower* (resp. *upper*) *semilattice* has $\inf\{x, y\}$ (resp. $\sup\{x, y\}$) for any two elements $x, y \in X$. One often writes $x \wedge y$, $x \vee y$.

Lattice. A poset that is simultaneously a lower and an upper semilattice.

Complete lattice. Every subset has a supremum and an infimum.

Theorem. *A poset is a complete lattice iff each subset has a supremum. Similarly with infima.*

Proof. Let us have, in $(X, \leq)$, all suprema. We will determine the infimum of an $M \subseteq X$. Set

$$N = \{x \mid M \subseteq \uparrow x\}, \ i = \sup N.$$

For every $y \in M$ we have $N \subseteq \downarrow y$ and hence $i \leq y$; thus, i is a lower bound of the set M. If $M \subseteq \uparrow x$ then $x \in N$ and hence $x \leq i$ so that $i = \inf M$.

Directed (sub)sets. $D \subseteq (X, \leq)$ is *directed*, if every finite $K \subseteq D$ has an upper bound in D. Note that in part. it has to be non-void.

(More exactly, *up-directed* as opposed to the *down-directed* defined with lower bounds.)

A note on notation. In complete lattices one often uses for suprema resp. infima the symbols $\bigvee$ resp. $\bigwedge$ (and sometimes also other symbols like, $\bigsqcup$, $\bigcup$ etc.), like in $\bigvee\{x \mid x \in M\}$, $\bigvee_{j \in J} x_j$, etc..

The symbols like $\bigvee$, $\bigwedge$ etc. are frequently viewed as operational symbols (similarly as the $a \vee b$, $a \wedge b$).

Further, the suprema $\bigvee$, $\vee$ are often referred to as *joins* and the infima $\bigwedge$, $\wedge$ are referred to as *meets*.

The symbols sup and inf are, typically, used in the case when they do not have to exist generally. This convention is fairly standard.

Two fixed-point theorems

Theorem. (Bourbaki) *Let* $(X, \leq)$ *have* $\perp$ *and let every chain in* X

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots$$

have a supremum. Let $f : X \to Y$ *preserve suprema of chains. Then* f *has a fixed point.*

Proof. Start with $x_0 = \perp$ and define x_n by setting $x_{n+1} = f(x_n)$. As $x_0 = \perp \leq x_1$ we obtain inductively that $x_{n+1} = f(x_n) \leq f(x_{n+1}) = x_{n+2}$ so that $x_0 \leq x_1 \leq \cdots \leq x_n \leq \cdots$. Consider $y = \sup x_n$. Then $f(y) = \sup f(x_n) = \sup x_{n+1} = y$ and y is a fixed point.

Note that the y from the proof is the *least fixed point of* f. If $f(z) = z$ we have $\perp \leq z$, $f(\perp) \leq f(z) = z$, and by induction $f(x_n) \leq z$.

Theorem. (Tarski – Knaster) *Every monotone mapping of a complete lattice into itself has a fixed point.*

Proof. Let L be a complete lattice and let $f : L \to L$ be monotone. Set $M = \{x \,|\, x \leq f(x)\}$ a $s = \sup M$. For $x \in M$ we have $x \leq s$ and hence $x \leq f(x) \leq f(s)$ so that $f(s)$ is an upper bound of the set M and we have

$$s \leq f(s),$$

and from the monotony, $f(s) \leq f(f(s))$ so that $f(s) \in M$. Therefore also

$$f(s) \leq s,$$

and hence finally $f(s) = s$.

Details. in Text:

Chapter I: 2,3,5,6

Chapter II: 1,2,3,7(without examples)