MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
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LECTURE 6 (March 22, 2023) APPLICATIONS OF BAIRE'S THEOREM: NON-DIFFERENTIABLE CONTINUOUS FUNCTIONS, TRANSCENDENTAL GROWTH RATES OF PERMUTATION CLASSES

• *Non-differentiable continuous functions.* Let $I := [0,1]$. By $C(I)$ we denote the set of all continuous functions from I to $\mathbb{R}$. Recall that for $x \in \mathbb{R}$ and $\delta > 0$,

$$P(x,\,\delta) = (x-\delta,\,x+\delta)\setminus\{x\} = (x-\delta,\,x)\cup(x,\,x+\delta)$$

is the deleted δ-neighborhood of x. In this lecture we prove the following theorem.

Theorem 1 (wild functions) *There exists a function f in $C(I)$ such that $\forall\, x \in I\ \forall\, \delta > 0$,*

$$\sup\left(\left\{\left|\frac{f(y)-f(x)}{y-x}\right| \;\middle|\; y \in P(x,\,\delta)\cap I\right\}\right) = +\infty\ .$$

Recall that $f\colon I \to \mathbb{R}$ is differentiable at $x \in I$ if there exists finite derivative $f'(x) \in \mathbb{R}$.

Exercise 2 *The function f in Theorem 1 is continuous on I but is not differentiable at any point of I.*

• *Four lemmas.* We prove Theorem 1 with the help of four lemmas.

Lemma 3 (1st lemma) *If* $f \in C(I)$ *has the property that* $\forall\, x \in I$,

$$\sup\left(\left\{\left|\frac{f(y)-f(x)}{y-x}\right| \,\middle|\, y \in I \setminus \{x\}\right\}\right) = +\infty$$

then f *has the property in Theorem 1. Hence the parameter* δ *in Theorem 1 is superfluous.*

Proof. We assume that $f \in C(I)$ has for every $x \in I$ the stated property. $\forall\, x \in I\ \forall\, \delta > 0$ the set

$$Q(x,\, \delta) := I \setminus U(x,\, \delta) = [0,\, 1] \setminus (x-\delta,\, x+\delta)$$

is compact (Exercise 4). Let $M(x,\delta)$ be the minimum value of the continuous function $y \mapsto \left|\frac{f(y)-f(x)}{y-x}\right| \geq 0$ on $Q(x,\delta)$. For every given $x \in I$ and $\delta > 0$ and every $c > M(x,\delta)$, by the assumption there is a $y \in I \setminus \{x\}$ such that

$$\left|\frac{f(y)-f(x)}{y-x}\right| > c\,.$$

But then $y \notin Q(x,\delta)$, thus $y \in P(x,\delta)$ and wee see that f has the property in Theorem 1. $\square$

Exercise 4 *Show that the set* $Q(x,\delta)$ *is compact.*

Exercise 5 *Why is the function* $y \mapsto \left|\frac{f(y)-f(x)}{y-x}\right|$ *continuous?.*

Recall that for any set X, the infinity-norm

$$\|f\|_\infty := \sup(\{|f(x)| \mid x \in X\})$$

on the set B of bounded functions $f\colon X \to \mathbb{R}$ makes B a MS

$$(B,\, \|f-g\|_\infty).$$

Exercise 6 *Show that this is a* MS.

Lemma 7 (2nd lemma) *Let* (M, d) *be a* MS, $(x_n) \subset M$ *be a sequence with* $\lim x_n = x_0 \in M$ *and let* (f_n), $f_n\colon M \to \mathbb{R}$, *be a sequence of functions converging in the norm* $\|\cdot\|_\infty$ *to a continuous function* $f\colon M \to \mathbb{R}$. *Then*

$$\lim f_n(x_n) = f(x_0)\ .$$

Proof. By the triangle inequality,

$$|f_n(x_n) - f(x_0)| \le |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)|\ .$$

For a given $\varepsilon > 0$, we can make the first $|\cdot|$ on the right side $< \varepsilon/2$ for $n \ge n_0$ due to the assumption that $\|f_n - f\|_\infty \to 0$. The same holds for the second $|\cdot|$ on the right side if $n \ge n_1$ due to Heine's definition of continuity of f at the point x_0. Hence $n \ge \max(\{n_0, n_1\}) \Rightarrow |f_n(x_n) - f(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

A *broken line* going through the points (a_0, b_0), (a_1, b_1), ..., (a_k, b_k) in $\mathbb{R}^2$ in this order, where $a_0 < a_1 < \cdots < a_k$, is the function $f\colon [a_0, a_k] \to \mathbb{R}$ which is on every interval $[a_{i-1}, a_i]$, $i = 1, 2, \ldots, k$, defined by

$$f(x) = \frac{(b_i - b_{i-1})(x - a_{i-1})}{a_i - a_{i-1}} + b_{i-1}$$

(thus $f(a_{i-1}) = b_{i-1}$ and $f(a_i) = b_i$). Its graph on the interval $[a_{i-1}, a_i]$ is the segment joining the points (a_{i-1}, b_{i-1}) and (a_i, b_i). We call these segments just *segments (of the broken line).*

Exercise 8 *Every broken line is a continuous function.*

The *slope* of a plane line given by the equation $y = ax + b$ is the number a. The *slope* of a segment is the slope of the line extending the segment. The *secant (line)* of a function $f\colon M \to \mathbb{R}$, $M \subset \mathbb{R}$, is a line going through two distinct points on the graph of f.

Lemma 9 (3rd lemma) *It is true that*

$$\begin{aligned}&\forall\, \varepsilon > 0\; \forall\, f \in C(I)\; \exists\, g \in C(I)\; \exists\, M > 0\\ &\big(\|f - g\|_\infty < \varepsilon \wedge (x,\, y \in I,\, x \neq y \Rightarrow\\ &\Rightarrow |(g(y) - g(x))/(y - x)| < M)\big)\ .\end{aligned}$$

In words, every function $f \in C(I)$ can be arbitrarily closely approximated by a function $g \in C(I)$ with bounded slopes of secant lines.

Proof. Let $f \in C(I)$ and let an $\varepsilon > 0$ be given. Since the interval I is compact, the function f is uniformly continuous (Exercise 10). Hence for every sufficiently large m and every $i = 0, 1, \ldots, m$ the implication

$$\tfrac{i}{m} \le x \le \tfrac{i+1}{m} \Rightarrow |f(\tfrac{i}{m}) - f(x)|,\, |f(\tfrac{i+1}{m}) - f(x)| < \tfrac{\varepsilon}{2}$$

holds. We draw the broken line g through the points $(i/m, f(i/m))$, $i = 0, 1, \ldots, m$. For g the above implication holds too and with the same m (Exercise 11). Thus

$$\forall\, x \in I\, \big(|f(x) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon\big)$$

(Exercise 12). This means that g has the first required property. By Exercise 13 we have that for every two distinct numbers $x, y \in I$,

$$\left|\frac{g(y) - g(x)}{y - x}\right| \le s$$

where s is the largest, in absolute value, slope of a segment of the broken line g. Hence g has also the second required property. $\square$

Exercise 10 *Why is any $f \in C(I)$ uniformly continuous?*

Exercise 11 *Show that the displayed implication holds for the broken line g.*

Exercise 12 *Prove the displayed inequality that $\forall\, x \dots$.*

Exercise 13 *Prove the inequality $\dots \le s$.*

Lemma 14 (4th lemma) *It is true that*

$$\forall\, \varepsilon > 0\, \forall\, T > 0\, \exists\, g \in C(I)\, \big(\|g\|_\infty < \varepsilon \wedge (x \in I \Rightarrow$$
$$\Rightarrow \exists\, y \in I \setminus \{x\}\, (|(g(y) - g(x))/(y - x)| > T))\big)\ .$$

In words, there exist a continuous and $\|\cdot\|_\infty$-small function g defined on I such that one can lead through every point on its graph a secant line with a large slope.

Proof. Let $\varepsilon > 0$ and $T > 0$ be given. We take a large even $m \in \mathbb{N}$ such that $2m\varepsilon/3 > T$, take the $m+1$ points in the plane

$$\big(i/m,\ (\varepsilon/3)(1 - (-1)^i\big),\ i = 0,\, 1\, \dots,\, m\ ,$$

and draw the broken line g through them. It starts in $(0,0)$, finishes in $(1,0)$ and consists of $m/2$ hills with height $2\varepsilon/3$ and bases of width $2/m$. Thus $\|g\|_\infty = 2\varepsilon/3 < \varepsilon$. Let a point u on the graph of g be given. We lead through it the secant line extending the segment containing u (if u lies in two segments, we choose any of them). In absolute value it has the slope larger than T because both sides of any hill have in absolute value slope $\frac{2\varepsilon/3}{1/m} = \frac{2m\varepsilon}{3} > T$. $\square$

• *Proof of Theorem 1.* We prove that there exists a continuous function $f\colon I \to \mathbb{R}$ that is not differentiable at any point of I.

Proof of Theorem 1. For $n \in \mathbb{N}$ we define the set

$$A_n := \left\{f \in C(I) \mid \exists\, x \in I\ \forall\, y \in I \setminus \{x\}\ \left(\left|\tfrac{f(y)-f(x)}{y-x}\right| \le n\right)\right\}\ .$$

We show that every set A_n is a sparse subset of the MS

$$(C(I),\ \|f-g\|_\infty)$$

and by this we will be done. Indeed, by Proposition 17 below this MS is complete and therefore by Baire's theorem there exists a function

$$f \in C(I) \setminus \bigcup_{n=1}^{\infty} A_n\ .$$

Thus f is continuous and has the property described in the first Lemma 3 and therefore, by this lemma, has the property in Theorem 1 and by Exercise 2 the function f is not differentiable at any point of I.

We show that every set $A_n \subset C(I)$ is closed and contains no ball, i.e., that for every ball $B(f,r)$ in the MS, $B(f,r) \not\subset A_n$. It follows from this that A_n is a sparse set (Exercise 15).

We prove that A_n is closed by showing its closedness to limits. Let $(f_k) \subset A_n$ be a sequence with $\lim_{k\to\infty} f_k = f \in C(I)$; we show that $f \in A_n$. Since $f_k \in A_n$, there is a number $x_k \in I$ such that for every $y \in I \setminus \{x_k\}$,

$$\left|\frac{f_k(y)-f_k(x_k)}{y-x_k}\right| \le n\ .$$

We know from *Mathematical Analysis 1* that (x_k) has a convergent subsequence with a limit in I. To simplify notation, we assume that already $\lim_{k\to\infty} x_k = x_0 \in I$. For every $y \in I \setminus \{x_0\}$ we have, by the property of the point x_k and the second Lemma 7, that

$$n \ge \lim_{k\to\infty} \left|\frac{f_k(y)-f_k(x_k)}{y-x_k}\right| = \left|\frac{f(y)-f(x_0)}{y-x_0}\right|$$

(non-strict inequalities are preserved in limits). The number x_0 therefore witnesses that $f \in A_n$ and A_n is a closed subset of the MS.

It remains to find in the given ball $B(f,r) \subset C(I)$ a point (i.e., a function) $g \in B(f,r) \setminus A_n$. We define it as $g = g_1 + g_2$ where we find the functions g_1 and g_2 using the third and fourth Lemma 9 and 14, respectively. First we use Lemma 9 and obtain a function $g_1 \in C(I)$ and a constant $M > 0$ such that $\|f - g_1\|_\infty < r/2$ and that all secants of the graph of g_1 have slope in absolute value $< M$. Then we use Lemma 14 and obtain a function $g_2 \in C(I)$ such that $\|g_2\|_\infty < r/2$ and that through every point on the graph of g_2 there goes a secant line with slope in absolute value $> M + n$. By the triangle inequality,

$$\|f-g\|_\infty \le \|f-g_1\|_\infty + \|g_2\|_\infty < r/2 + r/2 = r$$

and $g \in B(f,r)$. Let $x \in I$ be arbitrary. By the property of the function g_2 we take a $y \in I \setminus \{x\}$ such that $|\frac{g_2(y)-g_2(x)}{y-x}| > M + n$. Then

$$\begin{aligned}\left|\frac{g(y)-g(x)}{y-x}\right| &= \left|\frac{g_2(y)-g_2(x)}{y-x} + \frac{g_1(y)-g_1(x)}{y-x}\right| \\ &\ge \left|\frac{g_2(y)-g_2(x)}{y-x}\right| - \left|\frac{g_1(y)-g_1(x)}{y-x}\right| \\ &> (M+n) - M = n\end{aligned}$$

and $g \notin A_n$. On the first line we used the definition of g, on the second the inequality from Exercise 16 and on the third the properties of the functions g_1 and g_2. $\square$

Exercise 15 *Prove that every closed set X (in a* MS*) with empty interior (i.e., X contains no ball) is sparse.*

Exercise 16 *Prove that for every two real numbers a and b,*

$$|a-b| \geq |a| - |b| \ .$$

• *Completeness of the* MS *of continuous functions with the infinity-norm metric.*

Proposition 17 *The metric space*

$$(C(I),\ \|f-g\|_\infty)$$

is complete.

Proof. Let $(f_n) \subset C(I)$ be a Cauchy sequence in this MS, i.e.,

$$\forall\, \varepsilon > 0\ \exists\, m \left(n,\, n' \geq m \Rightarrow \|f_n - f_{n'}\|_\infty < \varepsilon\right) \ .$$

Then for every $x \in I$ the sequence $(f_n(x)) \subset \mathbb{R}$ is Cauchy, therefore convergent, and we can define

$$f(x) := \lim f_n(x) \ .$$

Thus we have a function $f\colon I \to \mathbb{R}$ with the property that $f_n \to f$ pointwisely. Let us prove the uniform convergence, i.e., that $\|f - f_n\|_\infty \to 0$. Let $x \in I$ and $\varepsilon > 0$ be given. We take an m (it is independent of x) such that the above displayed Cauchy condition holds with $\varepsilon/2$. Then we take a $k \geq m$ such that $|f_k(x) - f(x)| < \varepsilon/2$ and get that $n \geq m \Rightarrow$

$$|f_n(x) - f(x)| \leq |f_n(x) - f_k(x)| + |f_k(x) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \ .$$

Thus $\lim f_n = f$ in this MS.

It remains to show that f is continuous (i.e., is an element of this MS). Let an $x_0 \in I$ and an $\varepsilon > 0$ be given. We take an n_0 such that

$$n \geq n_0 \Rightarrow \|f - f_n\|_\infty \leq \varepsilon/2 \ .$$

We take a $\delta > 0$ such that

$$x \in U(x_0,\, \delta) \cap I \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| \leq \varepsilon/2$$

(we are using continuity of f_{n_0} at x_0). Then $\forall\, x \in U(x_0, \delta) \cap I$,

$$|f(x) - f(x_0)| \leq |f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

and f is continuous at x_0. $\Box$

Only now is the proof of Theorem 1 in fact complete.

• *An application of Baire's theorem in enumeration of permutations.* For $m \leq n$ in $\mathbb{N} = \{1, 2, \dots\}$ and two permutations (i.e., bijections) $\pi\colon [m] \to [m]$ and $\rho\colon [n] \to [n]$ we write $\pi \preceq \rho$, and say that π *is contained in* ρ, if there exist numbers $i_1 < i_2 < \cdots < i_m$ in $[n]$ such that

$$\forall\, j,\, k \in [m]\, \big(\pi(j) < \pi(k) \iff \rho(i_j) < \rho(i_k)\big) \ .$$

Let $\mathcal{S}$ be the set of all finite permutations $\pi\colon [n] \to [n]$ for n running in $\mathbb{N}$ and let $\mathcal{S}_n \subset \mathcal{S}$ be the $n!$-element set of permutations of $[n]$.

Exercise 18 *Show that* $(\mathcal{S}, \preceq)$ *is a non-strict partial order.*

We say that a set $X \subset \mathcal{S}$ is a *permutation class* if for every two permutations π and ρ,

$$\pi \preceq \rho \in X \Rightarrow \pi \in X \ .$$

In the last roughly 20 years, many results on enumeration of permutation classes X, i.e., on the counting functions of the form

$$n \mapsto |X \cap \mathcal{S}_n|$$

($|A|$ denotes the cardinality of a finite set A), were obtained. One of the basic ones is the next theorem.

Theorem 19 (A. Marcus and G. Tardos, 2004) *Let X be a permutation class. Then*

$$X \neq \mathcal{S} \Rightarrow \exists\, c > 1\; \forall\, n \left(|X \cap \mathcal{S}_n| \le c^n\right) .$$

In words, any permutation class, with the exception of the class of all permutations, grows only at most exponentially.

Exercise 20 *Let $\pi \in \mathcal{S}_2$ be the identical permutation $\pi(1) = 1$, $\pi(2) = 2$ and let X be any permutation class such that $\pi \notin X$. Show that then $|X \cap \mathcal{S}_n| \le 1$ for every n.*

By the Marcus–Tardos theorem, for every permutation class X different from $\mathcal{S}$ one can define its finite *growth rate*

$$c(X) := \limsup_{n \to \infty} |X \cap \mathcal{S}_n|^{1/n} \in [0,\, +\infty) .$$

For example, it is known that $c(\{\rho \in \mathcal{S} \mid \rho \not\succeq \pi\}) = 4$ for any $\pi \in \mathcal{S}_3$. In fact,

$$|X \cap \mathcal{S}_n| = \frac{1}{n+1}\binom{2n}{n}$$

for every n for any of these six permutations classes X.

For some time there was a conjecture that every growth rate of a permutation class is an algebraic number. It was refuted by the following result.

Theorem 21 (M. Albert and S. Linton, 2009) *There is a non-empty closed set* $A \subset [0, +\infty)$ *such that* A *has no isolated point and every element of* A *is the growth rate of a permutation class.*

As we saw in the lecture before the last lecture, by Baire's theorem such set A is uncountable. Consequently, there exist uncountably many transcendental (i.e., non-algebraic) growth rates of permutation classes.

Exercise 22 *How does it exactly follow from Baire's theorem that the above set* A *is uncountable?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to Mgr. J. Rondoš, Ph.D. by the end of the coming Monday by e-mail (jakub.rondos@gmail.com) solutions to the Exercises 2, 4, 15, 20 and 22.