MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
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LECTURE 2 (February 22, 2023) OSTROWSKI'S THEOREM. COMPACT METRIC SPACES.

• *Ostrowski's theorem.* On any field F we have the *trivial norm.* It is a function $\|\cdot\|$ with $\|0_F\| = 0$ and $\|x\| = 1$ for $x \neq 0_F$.

Exercise 1. *Prove that a trivial norm is a norm.*

From the usual absolute value of $|\cdot|$ to $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$, we get many other norms by exponentiation.

Exercise 2. *Prove that for any $c > 0$, $|\cdot|^c$ is a norm (on $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$) if and only if $c \le 1$. We will call this norm the modified absolute value.*

For $\alpha \in \mathbb{Q}$ and a prime p, the *canonical p-adic norm* $\|\cdot\|_p$ is defined by

$$\|\alpha\|_p := p^{-\mathrm{ord}_p(\alpha)}$$

− in the general p-adic norm $|\cdot|_p$ we set $c := 1/p$.

Exercise 3. *Let $M := \{2, 3, 5, 7, 11, \dots\} \cup \{\infty\}$ and $\|\cdot\|_\infty := |\cdot|$ (ordinary absolute value). Prove for every nonzero number $\alpha \in \mathbb{Q}$* the product formula

$$\prod_{p \in M} \|\alpha\|_p = 1\ .$$

Exercise 4. *Let $\|\cdot\|$ be a nontrivial norm on the field $\mathbb{Q}$. Prove that $\exists\, n \in \mathbb{N}\,\big(n \ge 2 \wedge \|n\| \neq 1\big)$.*

Exercise 5. *Prove that for every two coprime numbers $a, b \in \mathbb{Z}$ there exist numbers $c, d \in \mathbb{Z}$ such that*

$$ac + db = 1 \ .$$

Theorem 6 (A. Ostrowski, 1916). *Let $\|\cdot\|$ be a norm on the field of rational numbers $\mathbb{Q}$. Then one of the following three cases occurs.*

1. *It is a trivial norm.*
2. *There exists a real $c \in (0, 1]$ such that $\|x\| = |x|^c$.*
3. *There exists a real $c \in (0, 1)$ and a prime number p such that $\|x\| = |x|_p = c^{\mathrm{ord}_p(x)}$.*

Modified absolute values and p-adic norms are therefore the only non-trivial norms on the field of rational numbers.

Proof. Let $\|\cdot\|$ be a nontrivial norm. By Exercise 4, there exists $n \in \mathbb{N} \setminus \{1\}$ such that $\|n\| \neq 1$. We have two cases.

1. *There exists an $n \in \mathbb{N}$ such that $\|n\| > 1$.* We denote the smallest such n as n_0. Apparently $n_0 \geq 2$ a

$$1 \leq m < n_0 \Rightarrow \|m\| \leq 1 \ . \tag{1}$$

There exists a unique real number $c > 0$ such that

$$\|n_0\| = n_0^c \ . \tag{2}$$

Any $n \in \mathbb{N}$ can be expanded in the base n_0:

$$\begin{aligned} &n = a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_s n_0^s \text{ where} \\ &a_i,\ s \in \mathbb{N}_0,\ 0 \leq a_i < n_0 \ \text{ and } \ a_s \neq 0 \ . \end{aligned}$$

For $n_0 = 10$ we get the standard decimal notation. So

$$\begin{aligned}
\|n\| &= \|a_0 + a_1 n_0 + a_2 n_0^2 + \cdots + a_s n_0^s\| \\
&\overset{\Delta\text{-ineq. and multipl. of } \|\cdot\|}{\leq} \textstyle\sum_{j=0}^{s} \|a_j\| \cdot \|n_0\|^j \\
&\overset{\text{eq. (1) a (2)}}{\leq} \textstyle\sum_{j=0}^{s} n_0^{jc} \ \leq \ n_0^{sc} \sum_{i=0}^{\infty} (1/n_0^c)^i \\
&\overset{n_0^s \leq n}{\leq} n^c C, \ \text{where } C := \textstyle\sum_{i=0}^{\infty} (1/n_0^c)^i \ .
\end{aligned}$$

Hence

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| \leq C n^c \right) . \tag{3}$$

This inequality actually holds even with $C = 1$. For each $m, n \in \mathbb{N}$, the multiplicativity of the norm and inequality (3) give

$$\|n\|^m = \|n^m\| \leq C\,(n^m)^c = C\,(n^c)^m \ .$$

If we take the m-th root, we get $\|n\| \leq C^{1/m} n^c$. For $m \to \infty$ we have $C^{1/m} \to 1$. So indeed

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| \leq n^c \right) . \tag{4}$$

We similarly derive the converse inequality $\|n\| \geq n^c$, $n \in \mathbb{N}_0$. For every $n \in \mathbb{N}$ the above expression of the number n in the base n_0 gives that

$$n_0^{s+1} > n \geq n_0^s \ .$$

By the Δ-inequality we have that

$$\|n_0\|^{s+1} = \|n_0^{s+1}\| \leq \|n\| + \|n_0^{s+1} - n\| \ .$$

Hence

$$\begin{aligned}
\|n\| \quad &\geq \quad \|n_0\|^{s+1} - \|n_0^{s+1} - n\| \overset{\text{eq. (2) and (4)}}{\geq} n_0^{(s+1)c} - (n_0^{s+1} - n)^c \\
&\overset{n \geq n_0^s}{\geq} \quad n_0^{(s+1)c}(n_0^{s+1} - n_0^s)^c = n_0^{(s+1)c}\left(1 - \left(1 - \tfrac{1}{n_0}\right)^c\right) \\
&\overset{n_0^{s+1} > n}{\geq} \quad n^c C', \quad \text{where } \; C' := 1 - \left(1 - \tfrac{1}{n_0}\right)^c > 0 \, .
\end{aligned}$$

The trick with the m-th root gives again

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| \geq n^c\right)$$

and so

$$\forall\, n \in \mathbb{N}_0 \left(\|n\| = n^c\right) \, .$$

From the multiplicativity of the norm we get that $\|x\| = |x|^c$ for any $x \in \mathbb{Q}$. By Exercise 2 $c \in (0, 1]$. Thus case 2 of Ostrowski's theorem holds.

2. *The remaining case when for every $n \in \mathbb{N}$ there is an $\|n\| \leq 1$ and there exists an $n \in \mathbb{N}$ such that $\|n\| < 1$.* Let n_0 be the smallest such n, again $n_0 \geq 2$. We claim that $n_0 = p$ is a prime number. Indeed, if n_0 had a decomposition $n_0 = n_1 n_2$ with $n_i \in \mathbb{Z}$ and $1 < n_1, n_2 < n_0$, we would get the contradiction

$$1 > \|n_0\| = \|n_1 n_2\| = \|n_1\| \cdot \|n_2\| = 1 \cdot 1 = 1 \, ,$$

where we used the multiplikativity of the norm and that $\|m\| = 1$ for every $m \in \mathbb{N}$ with $1 \leq m < n_0$. We show that every prime number q with $q \neq p$ has the norm $\|q\| = 1$. For the contrary, let $q \neq p$ be another prime number with norm $\|q\| < 1$. We take a large $m \in \mathbb{N}$ such that $\|p\|^m, \|q\|^m < \frac{1}{2}$. By Exercise 5 there are integers a and b such that $aq^m + bp^m = 1$. Taking norms in this equality gives the contradiction:

$$1 = \|1\| = \|aq^m + bp^m\| \le \|a\|\cdot\|q\|^m + \|b\|\cdot\|p\|^m < 1\cdot\frac{1}{2} + 1\cdot\frac{1}{2} = 1\ .$$

Here we used the triangle inequality, the multiplicativity of the norm, and the fact that now $\|a\| \le 1$ for every $a \in \mathbb{Z}$.

Thus $\|q\| = 1$ for every prime number q different from p. From this, using multiplicativity of the norm and the decomposition of the non-zero fraction x in the product of powers of prime numbers, we get the expression

$$\begin{aligned} \|x\| &= \left\| \prod_{q=2,\,3,\,5,\,\dots} q^{\mathrm{ord}_q(x)} \right\| = \prod_{q=2,\,3,\,5,\,\dots} \|q\|^{\mathrm{ord}_q(x)} = \|p\|^{\mathrm{ord}_p(x)} \\ &= c^{\mathrm{ord}_p(x)},\ \text{where } c := \|p\| \in (0, 1)\ . \end{aligned}$$

Also $\|0\| = c^{\mathrm{ord}_p(0)} = c^{\infty} = 0$. We are in case 3 of Ostrowski's theorem. □

The preceding proof is taken from the book

N. Koblitz, *p-adic Numbers, p-adic Analysis, and Zeta-Functions*, Springer-Verlag, New York, 1984.

The book contains a lot of interesting information about the p-adic norm $\|\cdot\|_p$ and related p-adic analysis.

• *Compactness of sets in* MSs. First, we introduce limits of sequences in MSs. Let (M, d) be a MS, $(a_n) \subset M$ be a sequence of points in it and $a \in M$ be a point. We say that (a_n) has the *limit* a *(in* (M, d)*)* if

$$\forall\,\varepsilon\ \exists\, n_0 \left(n \ge n_0 \Rightarrow d(a_n,\, a) < \varepsilon\right)\ .$$

From now on $\varepsilon > 0$ is a real number and $n_0, n \in \mathbb{N}$. We write that $\lim a_n = a$ or $\lim_{n\to\infty} a_n = a$. If the sequence (a_n) has a limit, we say that it is *convergent*, otherwise it is *divergent*.

Let (M, d) be a MS and $X \subset M$, for example $X = M$. We say that the set X is *compact* if

$$\forall\,(a_n) \subset X\ \exists\,(a_{m_n})\ \exists\, a \in X\ \big(\lim_{n\to\infty} a_{m_n} = a\big)\ .$$

In words: every sequence of points in the set X has a convergent subsequence with limit in X. The MS (M, d) *is compact* when the set M is compact.

The Bolzano–Weierstrass theorem states that on the real axis, i.e., in the MS $(\mathbb{R}, |x - y|)$, every closed and bounded interval $X = [a, b]$ is a compact set. We will give a few examples of compact sets and compact MSs.

Exercise 7. *In every* MS *every finite set is compact.*

Exercise 8. *Is the real axis (with the metric $|x-y|$) a compact* MS*?*

Exercise 9. *Which other intervals on the real axis besides $[a, b]$ are compact sets?*

Exercise 10. *Let $X = [a, b] \times [c, d]$ be a rectangle in the plane, that is, in the Euclidean space $(\mathbb{R}^2, e_2)$. Prove that X is a compact set.*

Exercise 11. *Let (M, d) be* MS*, $A, B \subset M$ and let us briefly write „is c.“ instead of „is a compact set“. Determine which of* *the following implications holds*

$$\begin{aligned} &A \textit{ and } B \textit{ are c.} \Rightarrow A \cup B \textit{ is c.} \\ &A \textit{ and } B \textit{ are c.} \Rightarrow A \cap B \textit{ is c.} \\ &A \subset B \textit{ and } B \textit{ is c.} \Rightarrow A \textit{ is c.} \\ &A \textit{ and } B \textit{ are c.} \Rightarrow A \setminus B \textit{ is c.} \end{aligned}$$

• *We extend the maximum principle from the real axis to general* MSs. But first we need to introduce continuous maps between MSs. Let (M, d) and (N, e) be MSs and $f\colon M \to N$ be a map between them. We say that it is *continuous at* $a \in M$ if

$$\forall \varepsilon\, \exists \delta\, \forall x \in M\, \big(d(x,\, a) < \delta \Rightarrow e(f(x),\, f(a)) < \varepsilon\big)\ .$$

Here $\delta > 0$ is a real number. *A map f is continuous* if it is continuous at every point $a \in M$.

Exercise 12. *Let $f\colon M \to N$ be a map between* MSs *and $a \in M$ be a point. Prove Heine's definition of continuity, that is, prove the equivalence*

$$\begin{aligned} &f \textit{ is continuous at } a \iff \\ &\iff \forall\, (a_n) \subset M\, \big(\lim\, a_n = a \Rightarrow \lim\, f(a_n) = f(a)\big)\ . \end{aligned}$$

Theorem 13 (attaining extrema). *Let (M, d) be a* MS*,*

$$f\colon M \to \mathbb{R}$$

be a continuous function from M to the real axis, and $X \subset M$ be a nonempty compact set. Then

$$\exists\, a,\, b \in X\, \forall\, x \in X\, \big(f(a) \le f(x) \le f(b)\big)\ .$$

Thus the function f attains on the set X its smallest value $f(a)$ and its largest value $f(b)$.

Proof. First we show that the image $f[X] = \{f(x) \mid x \in X\}$ is a bounded subset of $\mathbb{R}$. If the set $f[X]$ were not bounded from above, we could take a sequence $(a_n) \subset X$ with $\lim f(a_n) = +\infty$, i.e., such that $\forall\, c\, \exists\, n_0 \big(n \geq n_0 \Rightarrow f(a_n) > c\big)$. By the assumption, (a_n) has a convergent subsequence (a_{m_n}) with $\lim a_{m_n} = a \in X$. By the continuity of f at a and Exercise 12, $\lim f(a_{m_n}) = f(a) \in \mathbb{R}$. But this is a contradiction because $\lim f(a_{m_n}) = +\infty$. Boundedness of $f[X]$ from below can be proved similarly.

So we can define the real numbers $A := \inf(f[X])$ and $B := \sup(f[X])$. By the definition of infimum, there exists a sequence $(a_n) \subset X$ such that $\lim f(a_n) = A$. By the assumption, (a_n) has a convergent subsequence (a_{m_n}) with $\lim a_{m_n} = a \in X$. By the continuity of f at a and Exercise 12, $\lim f(a_{m_n}) = f(a)$. At the same time, however, since subsequences preserve limits, $\lim f(a_{m_n}) = A$. Thus $f(a) = A$ and for every $x \in X$,

$$f(a) = A \leq f(x)$$

because A is the infimum of the set $f[X]$. Similarly, we find $b \in X$ such that $f(b) = B$, and similarly $f(b) = B \geq f(x)$ for every $x \in X$. □

• *Products of* MSs. For the MSs (M, d) and (N, e), we define their *product* $(M \times N, d \times e)$ so that $M \times N$ is the Cartesian product of the sets M and N and the $d \times e$ metric on it is given by

$$(d \times e)((a_1,\, a_2),\, (b_1,\, b_2)) := \sqrt{d(a_1,\, b_1)^2 + e(a_2,\, b_2)^2}\ .$$

Exercise 14. *Prove that the product of two* MSs *is a* MS.

Exercise 15. *Prove that the product of two Euclidean* MSs

$$(\mathbb{R}^m,\, e_m) \ \ \textit{and} \ \ (\mathbb{R}^n,\, e_n)$$

is (except for a formality in notation) the Euclidean MS

$$(\mathbb{R}^{m+n},\, e_{m+n})\ .$$

What is the "formality"?

• *Characterization of compact sets in Euclidean* MSs. We defined the ball $B(a,r)$ in a MS last time. A set $X \subset M$ in a MS (M,d) is *open* if

$$\forall\, a \in X\ \exists\, r\ \big(B(a,r) \subset X\big)\ .$$

Here $r > 0$ is a real number, the radius of the ball $B(a,r)$. X is *closed* if $M \setminus X$ is open. X is *bounded* if

$$\exists\, a \in M\ \exists\, r\ \big(X \subset B(a,\, r)\big)\ .$$

The *diameter* of the set X is, for $V := \{d(a,b) \mid a,b \in X\} \subset [0,+\infty)$, defined as

$$\mathrm{diam}(X) := \begin{cases} \sup(V) & \ldots\ \text{the set } V \text{ is bounded from above and} \\ +\infty & \ldots\ \text{the set } V \text{ is unbounded from above}\ . \end{cases}$$

Exercise 16. *Prove that any set X is bounded if and only if* $\mathrm{diam}(X) < +\infty$.

Exercise 17. *Prove that for any unbounded set X there is a sequence $(a_n) \subset X$ such that $m < n \Rightarrow d(a_m, a_n) > 1$.*

In the following two exercises we review basic properties of open and closed sets in a MS.

Exercise 18. *Let (M,d) be a* MS. *Then the following holds.*

1. *The sets $\emptyset$ and M are both open and closed.*

2. *Any finite intersection of open subsets of M is an open set and any finite union of closed subsets of M is a closed set.*
3. *Any union of open subsets of M is an open set and any intersection of closed subsets of M is a closed set.*

Exercise 19. *Let (M, d) be a* MS *and $X \subset M$. Then*

$$\text{the set } X \text{ is closed} \iff$$
$$\iff \forall\,(a_n) \subset X\; \forall\, a \in M\; \big(\lim a_n = a \Rightarrow a \in X\big)\;.$$

Theorem 20 (on compactness). *The following holds.*

1. *If $X \subset M$ is a compact set in a* MS *(M, d), then X is closed and bounded. The opposite implication does not in general hold, by Exercise 22.*
2. *If (M, d) and (N, e) are two compact* MSs*, then their product $(M \times N, d \times e)$ is a compact* MS.

Proof. 1. If X is not closed, then by Exercise 19 there exists a convergent sequence $(a_n) \subset X$ such that $\lim a_n = a \in M \setminus X$. This sequence does not have a convergent subsequence with limit in X, since each subsequence has limit a. When X is not bounded, we easily construct a sequence $(a_n) \subset X$ such that $m < n \Rightarrow d(a_m, a_n) > 1$ (Exercise 17). This sequence clearly has no convergent subsequence.

2. Let $(a_n) = ((a_{n,1}, a_{n,2}))$ be a sequence in the product MS. We choose a subsequence (b_n) such that $(b_{n,1})$ has a limit $b \in M$ in (M, d). From (b_n) we select a subsequence (c_n) such that $(c_{n,2})$ has a limit $c \in N$ in (N, e). It is not difficult to see that (c_n) is

a subsequence of the sequence (a_n) and that it has in the product MS the limit

$$\lim c_n = (b,\, c) \in M \times N\ .$$

$\square$

Exercise 21. *Let* (M, d) *be a compact* MS *and* $X \subset M$ *be a closed set. Prove that* X *is compact.*

Exercise 22. *Let* M *be an infinite set and the metric* d *on it is given as* $d(a, b) = 1$ *for* $a \neq b$ *and* $d(a, a) = 0$*. Show that* (M, d) *is a* MS *that is bounded and closed but not compact.*

Theorem 23 (compact sets in $\mathbb{R}^n$). *In every Euclidean* MS $(\mathbb{R}^n, e_n)$*,* $X \subset \mathbb{R}^n$ *is compact if and only if it is bounded and closed.*

Proof. By the first part of the previous theorem, it suffices to prove that every bounded and closed set $X \subset \mathbb{R}^n$ is compact. From its boundedness it follows that for a real number $a > 0$,

$$X \subset K := [-a,\, a]^n = [-a,\, a] \times [-a,\, a] \times \cdots \times [-a,\, a] \subset \mathbb{R}^n\ .$$

The Euclidean MS (K, e_n) is compact by the Bolzano–Weierstrass theorem, part 2 of the previous theorem, and Exercise 15. Clearly, X is also closed in (K, e_n) (problem 24), so according to Exercise 21, X is compact in (K, e_n) and therefore in $(\mathbb{R}^n, e_n)$ (Exercise 25). $\square$

Exercise 24. *Let* (M, d) *be a* MS*,* $A \subset B \subset M$ *and* A *be a closed set in* $(M, d) \Rightarrow A$ *is closed also in the subspace* (B, d)*.*

Exercise 25. *Let* (M, d) *be a* MS *and* $A \subset B \subset M$*. Then* A *is compact in* $(M, d) \iff A$ *is compact in the subspace* (B, d)*.*

THANK YOU FOR YOUR ATTENTION

Homework Exercises. Please send to Mgr. J. Rondoš, Ph.D. by the end of the coming Monday by e-mail (jakub.rondos@gmail.com) solutions to the Exercises 5, 9, 11, 17 and 22.