MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2024/25
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LECTURE 14 (May 21, 2025) SOME PARTICULAR DIFFERENTIAL EQUATIONS

• *Newton's law of force.* Differential equations (DE), relations between values of derivatives of the unknown function, are basic tools in mathematical models in physics, technology, biology, economics, etc. A basic example is *Newton's law of force*

$$mx'' = F\ ,$$

where $x = x(t) \in \mathbb{R}$ is the position at time t of a particle of mass m exposed to the force F (we consider only the simple one-dimensional case). More generally, the force can be a function of time, position of the particle and velocity: $F = F(t, x, x')$. In the simplest situation F is constant, or slightly more generally F depends only on t—then $x(t) = \int\int F$. This is the case, for example, in Earth's gravitational field which does not change in time and (for small scales) does not depend on the position of the particle and certainly not on its speed, which are all idealizations (especially the independence on x). *The Equation of Free Fall* is then

$$mx'' = -mg\ ,$$

where g is the gravitational acceleration constant. All its solutions are exactly the functions

$$X := \{x(t) = -\tfrac{1}{2}gt^2 + c_1t + c_2 \mid c_1,\ c_2 \in \mathbb{R}\}\ ,$$

where c_1 and c_2 are arbitrary constants. These express the fact that the movement of the falling particle is determined uniquely by its initial position $x(t_0)$ and velocity $x'(t_0)$ at some time instant t_0.

Exercise 1 *Prove that the solutions of The Free Fall Equation are precisely the functions in X. Can we determine the movement of the particle uniquely by the position $x(t_0)$ and velocity $x'(t_1)$ at different times t_0 and t_1?*

• *As a second example of* DE we present *The Radioactive Decay Equation*

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -kR \; .$$

It describes the evolution of the quantity $R = R(t)$ of decaying radioactive material in time t, where k is the material constant. It is clear that each function

$$R = R(t) = c \exp(-kt) \; ,$$

where c is a constant, is a solution to this equation.

DE branch in *ordinary differential equations* (ODE) which involve functions of just one variable, and *partial differential equations* (PDE) involving functions of several variables and their partial derivatives. Both previous equations are ODE. In the last and this lecture, we limit ourselves to ODE.

• *Before we completely leave* PDE*,* we mention for the sake of interest three important examples: *Laplace's Equation* (or *The Equation of Potential*)

$$u = u(x,\, y) \; : \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; ,$$

The Diffusion Equation (or *The Equation of Heat Conduction*)

$$u = u(x,\, t)\ :\quad \alpha^2 \cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

a *The Wave Equation*

$$u = u(x,\, t)\ :\quad a^2 \cdot \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\ ,$$

where α and a are constants. Physical meaning of these equations is apparent from their names.

• *The general form of an* ODE for the unknown function $y = y(x)$ is ($n \in \mathbb{N}$)

$$F(x,\, y,\, y',\, y'',\, \ldots,\, y^{(n)}) = 0\ ,$$

where F is a function in $n+2$ variables. The *order of the equation* is the highest order of derivative occurring in the equation. The above equation for the free fall is a second-order (ordinary differential) equation, while the radioactive decay equation is first-order.

A differential equation of the form ($n \in \mathbb{N}$)

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = b(x)$$

where $a_i(x)$ and $b(x)$ are given functions and $y = y(x)$ is an unknown function, is the *linear differential equation (of order n and with right side $b(x)$)*. For $b(x) = 0$ we speak of the *homogeneous linear differential equation.*

Differential equations that are not of this form (and therefore depend on some variables for the unknown function and its derivatives non-linearly), are *nonlinear differential equations.* For example *The Pendulum Equation*

$$\theta'' + (g/l)\sin\theta = 0\ ,$$

which describes the motion of a pendulum of length l swinging in a homogeneous gravitational field (g is the constant of gravitational acceleration) — the angle $\theta = \theta(t)$ is the deviation of the pendulum from the vertical at time t — is nonlinear. For small angles θ it holds that $\sin\theta \approx \theta$ and we can solve the linear approximation of the pendulum equation $\theta'' + (g/l)\theta = 0$, which is a linear ODE. The equation of free fall and the radioactive decay equation is linear.

Exercise 2 *Try to guess some solution to the equation*

$$\theta'' + (g/l)\theta = 0 \ .$$

• *Algebraic differential equations.* Differential equation (again $n \in \mathbb{N}$)

$$F(x,\, y,\, y',\, y'',\, \dots,\, y^{(n)}) = 0 \ ,$$

in which F is a polynomial in $n + 2$ variables, are *algebraic differential equations* (with the abbreviation ADE). Since the lecturer was and is interested in these equations, we now present (without proof) for the sake of interest three results about ADE. For the first of them, we recall that the *(Euler's gamma) function* $\Gamma(z)$ is defined for complex z with $\mathrm{re}(z) > 0$ by the integral

$$\Gamma(z) := \int_0^{+\infty} t^{z-1} e^{-t} \,\mathrm{dt} \ .$$

Exercise 3 *Show that for every* $z \in \mathbb{C}$ *s* $\mathrm{re}(z) > 0$ *this integral converges. The integrand needs to be estimated both at* 0 *and at* $+\infty$*.*

Exercise 4 *Compute that* $\Gamma(1) = 1$ *and prove that* $\Gamma(z)$ *satisfies the functional equation*

$$\Gamma(z + 1) = z\Gamma(z) \ .$$

Hint: integration by parts.

Thus $\Gamma(n+1) = n!$ for every $n \in \mathbb{N}_0$ and we see that the gamma function extends the factorial function. O. Hölder proved by means of the above functional equation the following theorem.

Theorem 5 (O. Hölder, 1887) *The gamma function does not satisfy any nontrivial* ADE, *for any nonzero complex polynomial F with $n+2$ variables.*

To state another result about ADE we define in the complex unit circle $|z| < 1$ the functions

$$\vartheta(z) := \sum_{n=0}^{\infty} z^{n^2} \ \text{ and } \ P(z) = \sum_{n=0}^{\infty} p(n) z^n := \prod_{n=1}^{\infty} \frac{1}{1 - z^n} \ .$$

Exercise 6 *Prove that the coefficients in the last power series are natural numbers, so $p(n) \in \mathbb{N}$ for every $n \in \mathbb{N}_0$, and that $p(n)$ is the number of* partitions of the number n, *the number of representations of n as a sum of natural numbers. Sums that differ only in the order of summands are not considered as different.*

Theorem 7 (positively about ADE) *Both $\vartheta(z)$ and $P(z)$ satisfy a non-trivial (and quite complicated)* ADE.

Finally, we preface the third result about ADE by mentioning that differential equations can be considered besides the domain of functions also in the domain of formal power series, which may have zero radius of convergence and do not define any function.

Exercise 8 *Consider the formal power series*

$$M(x) := \sum_{n=0}^{\infty} n! \cdot x^n = 1 + x + 2x^2 + 6x^3 + 24x^4 + \dots .$$

Derive a non-trivial ADE, *actually a first-order linear* DE *that* $M(x)$ *satisfies.*

We now define another formal power series

$$B(x) = \sum_{n=0}^{\infty} B_n x^n := \sum_{k=0}^{\infty} \frac{x^k}{(1-x)(1-2x)\dots(1-kx)} = 1 + \dots .$$

Next, we define for $k \in \mathbb{N}$

$$\sum_{n=1}^{\infty} S(n,k)x^n := \frac{x^k}{(1-x)(1-2x)\dots(1-kx)} .$$

It is clear that always $S(n,k) \in \mathbb{N}_0$. These numbers are called *Stirling numbers (of the second kind).*

Exercise 9 *Prove that for* $k, n \in \mathbb{N}$ *the number* $S(n,k)$ *is exactly the number of set partitions (disjoint unions of nonempty sets) of any* n*-element set in* k *blocks. Hint: the coefficient at* x^n *in the expansion of the rational function*

$$\frac{x^k}{(1-x)(1-2x)\dots(1-kx)}$$

counts words u *of length* n *over the alphabet* $[k] := \{1, 2, \dots, k\}$ *with the properties that (i) every* $i \in [k]$ *occurs in* u *and (ii) for each* $i, j \in [k]$ *with* $i < j$ *the first occurrence of* i *in* u *precedes the first occurrence of* j*.*

The above coefficients B_n thus express in terms of the Stirling numbers as

$$B_n = \sum_{k=1}^{n} S(n,\, k)$$

and B_n is the number of all set partitions of an n-element set. B_n are so-called *Bell numbers.*

Theorem 10 (M. Klazar, 2003) *The formal power series*

$$B(x) = \sum_{n=0}^{\infty} B_n x^n \ ,$$

that is, the ordinary generating function of Bell numbers, does not satisfy any non-trivial ADE.

The proof method is similar to Hölder's theorem. One proves that no non-trivial ADE is compatible with the following functional equation for $B(x)$.

Exercise 11 *Using the above definition of the power series $B(x)$, prove that*

$$B(x) = 1 + \frac{x}{1-x} \cdot B(x/(1-x)) \ .$$

Exercise 12 *Derive a nontrivial* ADE *for the exponential generating function of Bell numbers, which is*

$$\sum_{n=0}^{\infty} \frac{B_n x^n}{n!} = \mathrm{e}^{\mathrm{e}^x - 1} \ .$$

• ODE *with separated variables* is a general first-order nonlinear differential equation of the form

$$y(a) = b \wedge y' = f(x) \cdot g(y) \qquad \text{(SEP)}$$

for the unknown function $y = y(x)$ with the prescribed value $y(a) = b$ $(a, b \in \mathbb{R})$, where $f(x)$, resp. $g(y)$, is a function defined and continuous on some open interval $I \ni a$, resp. $J \ni b$, and $g \neq 0$ on J. We now locally uniquely solve this type of equation by the function $y\colon I' \to J$, for some open interval I' satisfying $a \in I' \subset I$. We will see that the solution is expressed (but only implicitly) using indefinite integrals of the functions $1/g$ and f.

We transform the equation in the form

$$\frac{y'}{g(y)} = f(x)$$

and rewrite it using a fixed function $G := \int 1/g$ (a primitive function to $1/g$ on the interval J) as

$$\forall\, x \in I' \left(G(y(x))' = f(x)\right) .$$

So we have the equation

$$\forall\, x \in I' \left(G(y(x)) = F(x) + c\right) ,$$

where $F := \int f$ is a given function, primitive to the function f on the interval I, and c is an (integration) constant. The solution $y(x)$ of the original equation (SEP) is thus given as an implicit function by the relation

$$\forall\, x \in I' \big(\underbrace{G(y(x)) = F(x) + c}_{(*)}\big),\ G = \int \frac{1}{g} \ \text{and}\ F = \int F .$$

The constant c is determined by the relation $G(b) = F(a) + c$. It follows from the implicit function theorem that there exists an open interval I' with $a \in I' \subset I$ and a uniquely determined function $y\colon I' \to J$ such that $y(a) = b$ and on I' the relation $(*)$ holds. So we have on I' a unique solution to the equation (SEP).

Exercise 13 *Explain the use of the implicit function theorem in this situation (for example, why its assumptions are met). But why is the solution of the equation* (SEP) *locally unique, when the primitive functions G and F are far from unique?*

Exercise 14 *Does not the local uniqueness of the solution of the equation* (SEP) *follow from Picard's theorem?*

• *Linear* ODE *of the 1st order.* We should be able to solve any linear differential equation of the 1st order, and this will conclude our course. It is an equation of the form ($x_0, y_0 \in \mathbb{R}$)

$$y(x_0) = y_0 \wedge y' + a(x)y = b(x) \ , \tag{LIN}$$

where $y = y(x)$ is an unknown function and the functions $a(x)$ and $b(x)$ are given, defined and continuous on some open interval $I \ni x_0$.

Exercise 15 *Does not the local uniqueness and existence of the solution to the equation* (LIN) *follow from Picard's theorem?*

Well, it does, so all we have to do is to solve the equation (that is, to express its solution in terms of the coefficients a and b by means of known functions and known operations). First we find a function $c = c(x)$, so-called *integration factor*, such that

$$c \cdot (y' + ay) = (cy)' \ .$$

Then $cy' + acy = cy' + c'y$ and c must satisfy the equation $ac = c'$, i.e. $(\log c)' = a$. The function $c = e^A$, where $A = \int a$, has the required property. We multiply the initial linear equation by the integration factor and get that

$$(cy)' = \underbrace{c(y' + ay) = cb}_{c \cdot (\mathrm{LIN})} \ .$$

So $(cy)' = cb$ and $cy = D + c_0$, where $D = \int cb$ and c_0 is an integration constant. So we have the solution $y = c^{-1}(D + c_0)$. To sum up,

$$y(x) = e^{-A(x)}\left(\int e^{A(x)}b(x) + c_0\right),\ A(x) = \int a(x)\ .$$

Note that $y(x)$ is defined on the whole I (the domain of definition of the functions a and b) and that each initial condition $y(x_0) = y_0$ corresponds to exactly one value of the integration constant c_0 for which it is satisfied.

Exercise 16 *Solve the equation with separated variables*

$$v \cdot v' = -\frac{gR^2}{(R+x)^2}\ .$$

Here $R \approx 6378\,\mathrm{km}$ *is the radius of the Earth,* $g \approx 9.81\,\mathrm{ms}^{-2}$ *is the gravitational acceleration,* $x > 0$ *is the height (in meters) of a particle that was ejected from Earth's surface with the speed* $v = v_0$*, and* $v = v(x)$ *is its speed at the height* x*. Calculate the* escape velocity *(also* the second cosmic velocity*), i.e., the velocity* v_0 *for which the particle will never fall back to Earth.*

Exercise 17 *Consider a particle with mass* m *that falls from the rest under the influence of constant gravity and on which, in addition to the weight, the resistance of the environment acts in such a way that the strength of the resistance is proportional to the speed of the particle. Find a 1st order linear* ODE *for this problem and solve it. Calculate the limit velocity that the particle (almost) reaches.*

THANK YOU FOR YOUR ATTENTION!

Here are twelve exam questions.

1. Define metric space and spherical metric. Prove that the hemisphere is not flat — T. 12 in L. 1.
2. Prove Ostrowski's theorem — T. 6 in L. 2.
3. Prove the Heine–Borel theorem — T. 11 in L. 3.
4. Prove the existence of n-th roots in $\mathbb{C}$ — T. 2 in L. 4.
5. Prove Baire's theorem — T. 19 in L. 4.
6. Explain, how to show that
$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$
— see L. 7.
7. Prove that the MS $C([0,1])$ of continuous functions (with the maximum metric) is complete — P. 17 in L. 6.
8. Prove the case $d = 2$ or the case $d = 3$ of Pólya's theorem — T. 8 in L 8.
9. Prove that $\rho \neq 0$ — T. 6 in L. 10.
10. Prove the Cauchy–Goursat theorem for rectangles — T. 12 in L. 10.
11. Prove Picard's theorem — T. 6 in L. 13.
12. Solve the differential equation $y' + ay = b$ for the unknown function $y = y(x)$ (and given functions $a(x)$ and $b(x)$) — see L. 14.