

MATHEMATICAL ANALYSIS 3 (NMAI056)

summer term 2024/25

lecturer: Martin Klazar

LECTURE 13 (May 14, 2025) EXISTENCE OF SOLUTIONS OF DIFFERENTIAL EQUATIONS: PICARD'S AND PEANO'S THEOREMS

- *One of the seven Millennium Problems* announced by the Clay Mathematical Institute in 2000 — by solving any of them one can earn 10^6 \$ — is the problem if there exists a smooth solution to the Navier–Stokes (partial differential) equations which describe flow of fluid in the (3-dim.) space.
- *Banach's fixed point theorem.* For Picard's theorem on differential equations, we will need two results about complete metric spaces, with which we therefore begin. The first of these is the well-known existence result of fixed points of a *contracting mapping* (*contraction*)

$$f: M \rightarrow M$$

of the metric space (M, d) into itself. This is any mapping such that for some constant $c \in (0, 1)$ for every $a, b \in M$,

$$d(f(a), f(b)) \leq c \cdot d(a, b)$$

— f contracts distances by some factor less than 100%.

Exercise 1 *Prove that every contracting mapping of a metric space into itself is continuous.*

Theorem 2 (Banach's on fixed point) *Every contraction*

$$f: M \rightarrow M$$

of a complete MS has a unique fixed point: a point $a \in M$ such that

$$f(a) = a .$$

Furthermore, each sequence $(a_n) \subset M$ of iterations of f , where the point $a_1 \in M$ is arbitrary and $a_n = f(a_{n-1})$ for $n > 1$, converges to this fixed point a .

Proof. We show that any sequence $(a_n) \subset M$ of iterations of the function f is Cauchy. This can be seen immediately from the estimate that for every two indices $m > n$ it holds that (c is a constant from the definition of the contraction)

$$\begin{aligned}
 d(a_m, a_n) & \stackrel{\Delta\text{-inequality}}{\leq} \sum_{i=n}^{m-1} d(\underbrace{a_{i+1}}_{f(a_i)}, a_i) \\
 & \stackrel{f \text{ is contr., def. of } a_i}{\leq} \sum_{i=n}^{m-1} c^{i-1} \cdot d(a_2, a_1) \\
 & \stackrel{\text{adding } \geq 0 \text{ terms}}{\leq} d(a_2, a_1) \sum_{i=n}^{\infty} c^{i-1} \\
 & \stackrel{\Sigma \text{ of geom. series}}{=} \frac{d(a_2, a_1) \cdot c^{n-1}}{1 - c} \rightarrow 0, \quad n \rightarrow \infty .
 \end{aligned}$$

Since (M, d) is a complete MP, we can define

$$a := \lim_{n \rightarrow \infty} a_n \in M .$$

Then due to the continuity of the function f (Exercise 1),

$$f(a) = f(\lim_{n \rightarrow \infty} a_n) = \lim_{n \rightarrow \infty} f(a_n) = \lim_{n \rightarrow \infty} a_{n+1} = a$$

and a is a fixed point of f . You can prove its uniqueness in the following Exercise 3. \square

Exercise 3 *Prove that the fixed point of any contraction of any MS is unique.*

Exercise 4 *Prove Banach's Fixed Point Theorem under the weaker assumption that only some n -th iteration*

$$f^{[n]} := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times } f} : M \rightarrow M$$

of the mapping f of M into itself is contracting.

- *Completeness of a certain function space.* We also need the following complete MS.

Proposition 5 (completeness of continuous functions) *For every two real numbers $a < b$, the metric space*

$$(C[a, b], d)$$

of continuous functions $f: [a, b] \rightarrow \mathbb{R}$ is complete with respect to the maximum metric

$$d(f, g) = \max_{a \leq x \leq b} |f(x) - g(x)| .$$

Proof. This is Prop. 17 in lecture 6. □

- *Picard's theorem* is the following theorem about the existence and uniqueness of the solution of a first-order ordinary differential equation with explicit first derivative.

Theorem 6 (Picard's) *Let $a, b \in \mathbb{R}$ and $F: \mathbb{R}^2 \rightarrow \mathbb{R}$ be a continuous function for which there exists a constant $M > 0$ such that for every three numbers $u, v, w \in \mathbb{R}$,*

$$|F(u, v) - F(u, w)| \leq M \cdot |v - w| .$$

Then there exists $\delta > 0$ and a uniquely defined function

$$f: [a - \delta, a + \delta] \rightarrow \mathbb{R} ,$$

such that

$$f(a) = b \wedge \forall x \in [a - \delta, a + \delta] \left(f'(x) = F(x, f(x)) \right) . \quad (1)$$

Proof. Let $I := [a - \delta, a + \delta]$, for some small $\delta > 0$ to be determined later. It is easy to see (Exercise 7) that the solvability of the equation (1) for the unknown function f is equivalent to the solvability of the equation

$$\forall x \in I \left(f(x) = b + \int_a^x F(t, f(t)) \, dt \right) , \quad (2)$$

also for the unknown function f . We show that for any sufficiently small $\delta > 0$, the equation (2), and therefore also the equation (1), has on the interval I a unique solution f . The right side of the equation (2) defines the map

$$A: C(I) \rightarrow C(I)$$

from the set of continuous functions $f: I \rightarrow \mathbb{R}$ into itself, namely $A(f) = g$

$$\text{where for } x \in I, g(x) := b + \int_a^x F(t, f(t)) \, dt .$$

We prove that A is a contraction of the MS $(C(I), d)$ with the maximum metric d into itself. By Theorem 2 and Proposition 5, A has a unique fixed point, the function $f \in C(I)$ such that $A(f) = f$, and equations (1) and (2) have unique solutions.

We prove that for any sufficiently small $\delta > 0$, A is a contraction. Let $f, g \in C(I)$. Then

$$\begin{array}{ll}
& d(A(f), A(g)) = \\
\stackrel{\text{def. of } d}{=} & \max_{x \in I} |A(f)(x) - A(g)(x)| \\
\stackrel{\text{def. of } A}{=} & \max_{x \in I} \left| \int_a^x F(t, f(t)) \, dt - \int_a^x F(t, g(t)) \, dt \right| \\
\stackrel{\text{linearity of } \int}{=} & \max_{x \in I} \left| \int_a^x (F(t, f(t)) - F(t, g(t))) \, dt \right| \\
\stackrel{|\int h| \leq \int |h|}{\leq} & \max_{x \in I} \int_a^x |F(t, f(t)) - F(t, g(t))| \, dt \\
\stackrel{\text{ass. on } F, h \leq j \Rightarrow \int h \leq \int j}{\leq} & \max_{x \in I} \int_a^x M |f(t) - g(t)| \, dt \\
\stackrel{h \leq j \Rightarrow \int h \leq \int j}{\leq} & \max_{x \in I} \int_a^x M \cdot d(f, g) \, dt \\
\stackrel{\int_a^x c = (x-a)c}{=} & \delta M \cdot d(f, g) .
\end{array}$$

For example, if $\delta = 1/2M$ then A is a contraction with the constant $c = 1/2$. \square

Exercise 7 *Prove that the function $f: I \rightarrow \mathbb{R}$ is a solution of the equation (1) if and only if f is a solution of the equation (2).*

For example, the equation

$$f(1) = -3 \wedge f' = f$$

has a unique solution on a neighborhood of 1, because here $F(u, v) = v$ and the condition on the function F is satisfied with the constant

$M = 1$. This solution is the function

$$f(t) = (-3/e) \exp(t) .$$

Exercise 8 *Prove that Picard's theorem holds even under this weaker assumption about the function F : there exist constants $h, M > 0$ such that*

$$F: (a - h, a + h) \times (b - h, b + h) \rightarrow \mathbb{R}$$

is continuous and for every two pairs (u, v) and (u, w) from the definition domain F ,

$$|F(u, v) - F(u, w)| \leq M \cdot |v - w| .$$

• *Peano's Theorem* is the following theorem about the existence (but no longer uniqueness) of solutions to differential equations of the same kind as above.

Theorem 9 (Peano's) *Let $(a, b) \in U \subset \mathbb{R}^2$, where U is an open set in the Euclidean plane \mathbb{R}^2 , and $F: U \rightarrow \mathbb{R}$ be a continuous function. Then there exists a $\delta > 0$ and a function*

$$f: [a - \delta, a + \delta] \rightarrow \mathbb{R}$$

such that

$$f(a) = b \wedge \forall x \in [a - \delta, a + \delta] \left(f'(x) = F(x, f(x)) \right) .$$

Proof. First, we note that it suffices to prove the version of Peano's theorem, let us call it VP2, in which the interval $[a - \delta, a + \delta]$ is replaced by the interval $[a, a + \delta]$. Indeed, by VP2 there exists a $\delta' > 0$ and a function f_1 such that

$$f_1(-a) = b \quad \forall t \in [-a, -a + \delta'] \left(f_1'(t) = G(t, f_1(t)) \right) ,$$

where $G(u, v) := -F(-u, v)$. Then for $f_2(t) := f_1(-t)$ we have $f_2(a) = b$ and for every $t \in [-\delta' + a, a]$,

$$f_2'(t) = -f_1'(-t) = -G(-t, f_1(-t)) = F(t, f_2(t)) .$$

We combine this solution of our problem to the left of a with some of its solutions to the right of a , obtained again according to VP2, and we get a solution on a two-sided δ -neighborhood of the point a (Exercise 10).

So we prove VP2: there exists a $\delta > 0$ and a function $f: [a, a + \delta] \rightarrow \mathbb{R}$ such that

$$f(a) = b \wedge \forall t \in [a, a + \delta] \left(f'(t) = F(t, f(t)) \right) .$$

We take constants $a', b' > 0$ such that F is defined and continuous on $[a, a + a'] \times [b - b', b + b']$. So $|F| < L$ on this set, for some constant $L > 0$. Let's take the interval

$$I := [a, a + c], \quad \text{where } c := \min(a', b'/L) ,$$

and the set \mathcal{A} of functions

$$\{f: I \rightarrow \mathbb{R} \mid f(a) = b \wedge (s, t \in I \Rightarrow |f(s) - f(t)| \leq L|s - t|)\} .$$

According to the choice of c , for each $f \in \mathcal{A}$ the composite function $F(t, f(t))$, $t \in I$, is well-defined, continuous and bounded (by the constant L). We can therefore define the functional $P: \mathcal{A} \rightarrow [0, +\infty)$,

$$P(f) := \max_{t \in I} \left| f(t) - b - \int_a^t F(s, f(s)) \, ds \right| ; .$$

It is easy to see as before that if $P(f) = 0$, then f is a solution of VP2: $f(a) = b$ a $f'(t) = F(t, f(t))$ on $[a, a + c]$. According to the

following Theorem 14,

$$\mathcal{A} \subset C(I)$$

is a compact set in the MS $(C(I), d)$ with the maximum metric. It is easy to see that the functional P is continuous (Exercise 11), and therefore attains its minimum value on some function $\varphi \in \mathcal{A}$.

We show that $P(\varphi) = 0$ by finding functions $f_n \in \mathcal{A}$ for $n = 2, 3, \dots$ such that $P(f_n) \rightarrow 0$. We define them recursively as

$$\forall t \in [a, a + c/n] (f_n(t) := b)$$

and

$$\forall t \in (a + c/n, a + c) (f_n(t) := b + \int_a^{t-c/n} F(s, f_n(s)) \, ds) .$$

It is not difficult to see that this recursion correctly and uniquely defines the function f_n and that $f_n \in \mathcal{A}$ (Exercise 12). But then for each $t \in [a, a + c/n]$ we have that (according to the first part of the definition of f_n)

$$\left| f_n(t) - b - \int_a^t F(s, f_n(s)) \, ds \right| = \left| \int_a^t F(s, f_n(s)) \, ds \right| \leq \frac{Lc}{n}$$

and for each $t \in (a + c/n, a + c]$ that (according to the second part of the definition of f_n and by linearity of the integral)

$$\left| f_n(t) - b - \int_a^t F(s, f_n(s)) \, ds \right| = \left| \int_{t-c/n}^t F(s, f_n(s)) \, ds \right| \leq \frac{Lc}{n} ,$$

by means of simple ML integral estimates. Thus $0 \leq P(f_n) \leq \frac{Lc}{n}$ and indeed $P(f_n) \rightarrow 0$ for $n \rightarrow \infty$. \square

This proof is taken from: R. L. Pouso, Peano's Existence Theorem revisited, arXiv:1202.1152v1, 2012.

Exercise 10 *Explain in detail how the solution to the problem to the left of the point a can be combined with the solution to the right of point a and why these solutions can be combined.*

Exercise 11 *Prove that the functional P in the previous proof is continuous.*

Exercise 12 *Prove that the functions f_n in the previous proof are well defined and lie in the set \mathcal{A} .*

Exercise 13 (non-unique solutions) *For real numbers $a < 0 < b$ we define the function $f = f_{a,b}: \mathbb{R} \rightarrow \mathbb{R}$ as*

$$t \leq a \Rightarrow f(t) := (t - a)^3, \quad a \leq t \leq b \Rightarrow f(t) := 0$$

and

$$t \geq b \Rightarrow f(t) = (t - b)^3.$$

Prove that each of these functions is on \mathbb{R} a solution of the equation

$$f(0) = 0 \wedge f'(t) = 3f(t)^{2/3} := 3(f(t)^{1/3})^2.$$

The power $x^{1/3}$ is defined here for $x < 0$ as $-(-x)^{1/3}$.

Theorem 14 (Arzelà–Ascoli) *Let $I = [a, b]$ be a compact real interval and $C(I)$ be the MS of continuous functions $f: I \rightarrow \mathbb{R}$ with the maximum metric. A set $X \subset C(I)$ is compact if and only if*

$$\exists c > 0 \forall f \in X \forall x \in I (|f(x)| < c)$$

– the functions in X are uniformly bounded – and

$$\begin{aligned} &\forall \varepsilon > 0 \exists \delta > 0 \forall f \in X \forall x, y \in I \\ &(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon) \end{aligned}$$

– *the functions in X are uniformly uniformly continuous.*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Monday solutions to the Exercises 1, 3, 7 and 13. This is the last set of HWs.