MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
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**LECTURE 12 (May 3, 2023)** EXISTENCE THEOREMS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS: PICARD'S AND PEANO'S

• *One of the seven Millennium Problems* announced by the Clay Mathematical Institute in 2000 — by solving any of them one can earn $10^6$ \$ — is the problem if there exists a smooth solution to the Navier–Stokes (partial differential) equations which describe flow of fluid in the (3-dim.) space.

• *Banach's fixed point theorem.* For Picard's theorem on differential equations, we will need two results about complete metric spaces, with which we therefore begin. The first of these is the well-known existence result of fixed points of a *contracting mapping* (*contraction*)

$$f\colon M \to M$$

of the metric space $(M, d)$ into itself. This is any mapping such that for some constant $c \in (0, 1)$ for every $a, b \in M$,

$$d(f(a),\, f(b)) \le c \cdot d(a,\, b)$$

– $f$ contracts distances by some factor less than 100%.

**Exercise 1** *Prove that every contracting mapping of a metric space into itself is continuous.*

**Theorem 2 (Banach's on fixed point)** *Every contraction*

$$f\colon M \to M$$

*of a complete* MS *has a unique fixed point: a point $a \in M$ such that*

$$f(a) = a\ .$$

*Furthermore, each sequence $(a_n) \subset M$ of iterations of $f$, where the point $a_1 \in M$ is arbitrary and $a_n = f(a_{n-1})$ for $n > 1$, converges to this fixed point $a$.*

**Proof.** We show that any sequence $(a_n) \subset M$ of iterations of the function $f$ is Cauchy. This can be seen immediately from the estimate that for every two indices $m > n$ it holds that ($c$ is a constant from the definition of the contraction)

$$\begin{aligned} d(a_m,\, a_n) &\overset{\Delta\text{-inequality}}{\leq} \sum_{i=n}^{m-1} d(\underbrace{a_{i+1}}_{f(a_i)},\, a_i) \\ &\overset{f \text{ is contr., def. of } a_i}{\leq} \sum_{i=n}^{m-1} c^{i-1} \cdot d(a_2,\, a_1) \\ &\overset{\text{adding} \geq 0 \text{ terms}}{\leq} d(a_2,\, a_1) \sum_{i=n}^{\infty} c^{i-1} \\ &\overset{\sum \text{ of geom. series}}{=} \frac{d(a_2,\, a_1) \cdot c^{n-1}}{1-c} \to 0, \quad n \to \infty\ . \end{aligned}$$

Since $(M, d)$ is a complete MP, we can define

$$a := \lim_{n \to \infty} a_n \in M\ .$$

Then due to the continuity of the function $f$ (Exercise 1),

$$f(a) = f(\lim_{n\to\infty} a_n) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} a_{n+1} = a$$

and $a$ is a fixed point of $f$. You can prove its uniqueness in the following Exercise 3. $\square$

**Exercise 3** *Prove that the fixed point of any contraction of any* MS *is unique.*

**Exercise 4** *Prove Banach's Fixed Point Theorem under the weaker assumption that only some $n$-th iteration*

$$f^{[n]} := \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times } f} \colon M \to M$$

*of the mapping $f$ of $M$ into itself is contracting.*

• *Completeness of a certain function space.* We also need the following complete MS.

**Proposition 5 (completeness of continuous functions)** *For every two real numbers $a < b$, the metric space*

$$(C[a,\, b],\, d)$$

*of continuous functions $f \colon [a, b] \to \mathbb{R}$ is complete with respect to the maximum metric*

$$d(f,\, g) = \max_{a \le x \le b} |f(x) - g(x)| \;.$$

**Proof.** This is Prop. 17 in lecture 6. □

• *Picard's theorem* is the following theorem about the existence and uniqueness of the solution of a first-order ordinary differential equation with explicit first derivative.

**Theorem 6 (Picard's)** *Let $a, b \in \mathbb{R}$ and $F \colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function for which there exists a constant $M > 0$ such that for every three numbers $u, v, w \in \mathbb{R}$,*

$$|F(u,\, v) - F(u,\, w)| \le M \cdot |v - w| \;.$$

*Then there exists* $\delta > 0$ *and a uniquely defined function*

$$f\colon [a-\delta,\, a+\delta] \to \mathbb{R}\ ,$$

*such that*

$$f(a) = b \wedge \forall\, x \in [a-\delta,\, a+\delta]\, \big(f'(x) = F(x,\, f(x))\big)\ . \tag{1}$$

**Proof.** Let $I := [a-\delta,\, a+\delta]$, for some small $\delta > 0$ to be determined later. It is easy to see (Exercise 7) that the solvability of the equation (1) for the unknown function $f$ is equivalent to the solvability of the equation

$$\forall\, x \in I\, \big(f(x) = b + \int_a^x F(t,\, f(t))\; \mathrm{d}t\big)\ , \tag{2}$$

also for the unknown function $f$. We show that for any sufficiently small $\delta > 0$, the equation (2), and therefore also the equation (1), has on the interval $I$ a unique solution $f$. The right side of the equation (2) defines the map

$$A\colon C(I) \to C(I)$$

from the set of continuous functions $f\colon I \to \mathbb{R}$ into itself, namely $A(f) = g$

$$\text{where for } x \in I,\ g(x) := b + \int_a^x F(t,\, f(t))\; \mathrm{dt}\ .$$

We prove that $A$ is a contraction of the MS $(C(I), d)$ with the maximum metric $d$ into itself. By Theorem 2 and Proposition 5, $A$ has a unique fixed point, the function $f \in C(I)$ such that $A(f) = f$, and equations (1) and (2) have unique solutions.

We prove that for any sufficiently small $\delta > 0$, $A$ is a contraction. Let $f, g \in C(I)$. Then

$$
\begin{aligned}
& d(A(f),\, A(g)) = \\
\overset{\text{def. of } d}{=} \quad & \max_{x\in I} |A(f)(x) - A(g)(x)| \\
\overset{\text{def. of } A}{=} \quad & \max_{x\in I} \left| \int_a^x F(t,\, f(t))\ \mathrm{dt} - \int_a^x F(t,\, g(t))\ \mathrm{dt} \right| \\
\overset{\text{linearity of } \int}{=} \quad & \max_{x\in I} \left| \int_a^x \big(F(t,\, f(t)) - F(t,\, g(t))\big)\ \mathrm{dt} \right| \\
\overset{|\int h| \le \int |h|}{\le} \quad & \max_{x\in I} \int_a^x \big|F(t,\, f(t)) - F(t,\, g(t))\big|\ \mathrm{dt} \\
\overset{\text{ass. on } F,\ h \le j \Rightarrow \int h \le \int j}{\le} \quad & \max_{x\in I} \int_a^x M|f(t) - g(t)|\ \mathrm{dt} \\
\overset{h \le j \Rightarrow \int h \le \int j}{\le} \quad & \max_{x\in I} \int_a^x M \cdot d(f,\ ,g)\ \mathrm{dt} \\
\overset{\int_a^x c = (x-a)c}{=} \quad & \delta M \cdot d(f,\, g)\ .
\end{aligned}
$$

For example, if $\delta = 1/2M$ then $A$ is a contraction with the constant $c = 1/2$. $\square$

**Exercise 7** *Prove that the function $f\colon I \to \mathbb{R}$ is a solution of the equation* (1) *if and only if $f$ is a solution of the equation* (2)*.*

For example, the equation

$$f(1) = -3 \wedge f' = f$$

has a unique solution on a neighborhood of 1, because here $F(u, v) = v$ and the condition on the function $F$ is satisfied with the constant

$M = 1$. This solution is the function

$$f(t) = (-3/e)\exp(t)\ .$$

**Exercise 8** *Prove that Picard's theorem holds even under this weaker assumption about the function $F$: there exist constants $h, M > 0$ such that*

$$F\colon (a-h,\ a+h) \times (b-h,\ b+h) \to \mathbb{R}$$

*is continuous and for every two pairs $(u, v)$ and $(u, w)$ from the definition domain $F$,*

$$|F(u,\ v) - F(u,\ w)| \leq M \cdot |v - w|\ .$$

• *Peano's Theorem* is the following theorem about the existence (but no longer uniqueness) of solutions to differential equations of the same kind as above.

**Theorem 9 (Peano's)** *Let $(a, b) \in U \subset \mathbb{R}^2$, where $U$ is an open set in the Euclidean plane $\mathbb{R}^2$, and $F\colon U \to \mathbb{R}$ be a continuous function. Then there exists a $\delta > 0$ and a function*

$$f\colon [a-\delta,\ a+\delta] \to \mathbb{R}$$

*such that*

$$f(a) = b \wedge \forall\, x \in [a-\delta,\ a+\delta]\left(f'(x) = F(x,\ f(x))\right)\ .$$

**Proof.** First, we note that it suffices to prove the version of Peano's theorem, let us call it VP2, in which the interval $[a-\delta, a+\delta]$ is replaced by the interval $[a, a+\delta]$. Indeed, by VP2 there exists a $\delta' > 0$ and a function $f_1$ such that

$$f_1(-a) = b\ \text{ a }\ \forall\, t \in [-a, -a+\delta']\left(f_1'(t) = G(t,\ f_1(t))\right)\ ,$$

where $G(u,v) := -F(-u,v)$. Then for $f_2(t) := f_1(-t)$ we have $f_2(a) = b$ and for every $t \in [-\delta' + a, a]$,

$$f_2'(t) = -f_1'(-t) = -G(-t,\ f_1(-t)) = F(t,\ f_2(t))\ .$$

We combine this solution of our problem to the left of $a$ with some of its solutions to the right of $a$, obtained again according to VP2, and we get a solution on a two-sided $\delta$-neighborhood of the point $a$ (Exercise 10).

So we prove VP2: there exists a $\delta > 0$ and a function $f\colon [a, a+\delta] \to \mathbb{R}$ such that

$$f(a) = b \wedge \forall\, t \in [a,\ a+\delta]\,\big(f'(t) = F(t,\ f(t))\big)\ .$$

We take constants $a', b' > 0$ such that $F$ is defined and continuous on $[a, a+a'] \times [b-b', b+b']$. So $|F| < L$ on this set, for some constant $L > 0$. Let's take the interval

$$I := [a,\ a+c],\ \text{ where }\ c := \min(a',\ b'/L)\ ,$$

and the set $\mathcal{A}$ of functions

$$\{f\colon I \to \mathbb{R} \mid f(a) = b \wedge (s,\ t \in I \Rightarrow |f(s) - f(t)| \le L|s-t|)\}\ .$$

According to the choice of $c$, for each $f \in \mathcal{A}$ the composite function $F(t, f(t))$, $t \in I$, is well-defined, continuous and bounded (by the constant $L$). We can therefore define the functional $P\colon \mathcal{A} \to [0, +\infty)$,

$$P(f) := \max_{t \in I} \left| f(t) - b - \int_a^t F(s,\ f(s))\ \mathrm{ds} \right|\ ;.$$

It is easy to see as before that if $P(f) = 0$, then $f$ is a solution of VP2: $f(a) = b$ a $f'(t) = F(t, f(t))$ on $[a, a+c]$. According to the following Theorem 14,

$$\mathcal{A} \subset C(I)$$

is a compact set in the MS $(C(I), d)$ with the maximum metric. It is easy to see that the functional $P$ is continuous (Exercise 11), and therefore attains its minimum value on some function $\varphi \in \mathcal{A}$.

We show that $P(\varphi) = 0$ by finding functions $f_n \in \mathcal{A}$ for $n = 2, 3, \ldots$ such that $P(f_n) \to 0$. We define them recursively as

$$\forall\, t \in [a,\, a + c/n]\big(f_n(t) := b\big)$$

and

$$\forall\, t \in (a + c/n,\, a + c)\ \Big(f_n(t) := b + \int_a^{t-c/n} F(s,\, f_n(s))\ \mathrm{ds}\Big)\ .$$

It is not difficult to see that this recursion correctly and uniquely defines the function $f_n$ and that $f_n \in \mathcal{A}$ (Exercise 12). But then for each $t \in [a, a + c/n]$ we have that (according to the first part of the definition of $f_n$)

$$\left| f_n(t) - b - \int_a^t F(s,\, f_n(s))\ \mathrm{ds} \right| = \left| \int_a^t F(s,\, f_n(s))\ \mathrm{ds} \right| \le \frac{Lc}{n}$$

and for each $t \in (a + c/n, a + c]$ that (according to the second part of the definition of $f_n$ and by linearity of the integral)

$$\left| f_n(t) - b - \int_a^t F(s,\, f_n(s))\ \mathrm{ds} \right| = \left| \int_{t-c/n}^t F(s,\, f_n(s))\ \mathrm{ds} \right| \le \frac{Lc}{n}\ ,$$

by means of simple ML integral estimates. Thus $0 \le P(f_n) \le \frac{Lc}{n}$ and indeed $P(f_n) \to 0$ for $n \to \infty$. $\square$

This proof is taken from: R. L. Pouso, Peano's Existence Theorem revisited, arXiv:1202.1152v1, 2012.

**Exercise 10** *Explain in detail how the solution to the problem to the left of the point $a$ can be combined with the solution to the right of point $a$ and why these solutions can be combined.*

**Exercise 11** *Prove that the functional $P$ in the previous proof is continuous.*

**Exercise 12** *Prove that the functions $f_n$ in the previous proof are well defined and lie in the set $\mathcal{A}$.*

**Exercise 13 (non-unique solutions)** *For real numbers $a < 0 < b$ we define the function $f = f_{a,b}\colon \mathbb{R} \to \mathbb{R}$ as*

$$t \le a \Rightarrow f(t) := (t-a)^3,\ a \le t \le b \Rightarrow f(t) := 0$$

*and*

$$t \ge b \Rightarrow f(t) = (t-b)^3 \ .$$

*Prove that each of these functions is on $\mathbb{R}$ a solution of the equation*

$$f(0) = 0 \wedge f'(t) = 3f(t)^{2/3} := 3(f(t)^{1/3})^2 \ .$$

*The power $x^{1/3}$ is defined here for $x < 0$ as $-(-x)^{1/3}$ .*

**Theorem 14 (Arzelà–Ascoli)** *Let $I = [a,b]$ be a compact real interval and $C(I)$ be the* MS *of continuous functions $f\colon I \to \mathbb{R}$ with the maximum metric. A set $X \subset C(I)$ is compact if and only if*

$$\exists\, c > 0\ \forall\, f \in X\ \forall\, x \in I\ \big(|f(x)| < c\big)$$

*– the functions in $X$ are uniformly bounded – and*

$$\forall\, \varepsilon > 0\ \exists\, \delta > 0\ \forall\, f \in X\ \forall\, x,\, y \in I$$
$$\big(|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon\big)$$

– *the functions in $X$ are uniformly uniformly continuous.*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to Mgr. J. Rondoš, Ph.D. by the end of the coming Monday by e-mail (jakub.rondos@gmail.com) solutions to the Exercises 1, 3, 7 and 13.