MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2022/23
lecturer: Martin Klazar

LECTURE 10 (April 19, 2023) INTRODUCTION TO COMPLEX ANALYSIS 2

• *The Cauchy–Goursat thm. for rectangles and linear functions.* We continue with the proofs of the theorem on analyticity of integral functions and of Liouville's theorem; we presented both theorems in the last lecture. For $k \in \mathbb{N}$ and a segment $u \subset \mathbb{C}$, by its k-*equipartition* we mean the partition of u into k subsegments of the same length $|u|/k$. It is the image of the partition $0 < \frac{1}{k} < \frac{2}{k} < \dots < \frac{k-1}{k} < 1$ of the unit interval $I = [0, 1]$.

Exercise 1 *Let $a, b, \alpha, \beta \in \mathbb{C}$ and $a \neq b$. Prove from the definition of the integral that*

$$\int_{ab} (\alpha z + \beta) = \alpha \left(\frac{b^2}{2} - \frac{a^2}{2} \right) + \beta(b - a) = g(b) - g(a) \ ,$$

where $g(z) := \alpha z^2/2 + \beta z$. Hint: use equipartitions of the segment ab.

Corollary 2 (the easy C.–G. thm.) *Let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$, and $R \subset \mathbb{C}$ be a rectangle. Then*

$$\int_{\partial R} (\alpha z + \beta) = 0 \ .$$

Proof. Let a, b, c, d be the canonical vertices of the rectangle R and let $f(z) := \alpha z + \beta$. According to the definition of $\int_{\partial R}$ and of the previous problem, it is

$$\int_{\partial R} f = g(b) - g(a) + g(c) - g(b) + g(d) - g(c) + g(a) - g(d) = 0 \ .$$

$\square$

Proposition 3 ($\int_u$ and (R) $\int$) *Let $a, b \in \mathbb{C}$, $a \neq b$, $f\colon ab \to \mathbb{C}$ be a continuous function and $\varphi(t) := t(b-a) + a\colon [0,1] \to \mathbb{C}$ be the parametrization defining the segment $u = ab$. Then*

$$\begin{aligned} \int_u f &= \int_0^1 f(\varphi(t)) \cdot \varphi'(t) \ \mathrm{dt} = (b-a) \int_0^1 f(\varphi(t)) \ \mathrm{dt} \\ &= (b-a)\Big(\int_0^1 \mathrm{re}\big(f(\varphi(t))\big) \ \mathrm{dt} + i \cdot \int_0^1 \mathrm{im}\big(f(\varphi(t))\big) \ \mathrm{dt}\Big) \end{aligned}$$

(except for the first integral, all others are Riemann).

Exercise 4 *Prove the previous statement.*

For completeness, we give the definition of the integral $\int_\varphi f$ along the curve φ, fundamental for complex analysis. When

$$f\colon U \to \mathbb{C} \ \text{ is a function and } \ \varphi\colon [a,\, b] \to U$$

is a continuous and piece-wise smooth function, then we define the integral of the function f along the curve φ as

$$\begin{aligned} \int_\varphi f &:= \int_a^b f(\varphi(t)) \cdot \varphi'(t) \ \mathrm{dt} \\ &= \int_a^b \mathrm{re}\big(f(\varphi(t)) \cdot \varphi'(t)\big) \ \mathrm{dt} + i \cdot \int_a^b \mathrm{im}\big(f(\varphi(t)) \cdot \varphi'(t)\big) \ \mathrm{dt} \ , \end{aligned}$$

if the last two (real) Riemann integrals exist. Thus, our „line integral“ $\int_u$ is a special case of the curve integral $\int_\varphi$ according to Proposition 3.

Exercise 5 *Let* $\varphi(t)\colon [0, \frac{1}{2}] \to \mathbb{C}$,

$$\varphi(t) := \mathrm{e}^{2\pi it} = \sum_{n\geq 0} \frac{(2\pi it)^n}{n!}\ ,$$

be a parametrization of the upper unit semicircle, and $f(z) := z^2$. *Find*

$$\int_\varphi f\ .$$

Hint: follow the first line of the definition of $\int_\varphi$.

• *The constant* $\rho = 2\pi i$. The following theorem is an underappreciated pillar of complex analysis: if the constant ρ in it were 0, the Cauchy formulas we present next would not hold and the complex analysis would collapse.

Theorem 6 (the constant ρ) *Let* S *be the square with the vertices* $\pm 1 \pm i$. *Then*

$$\rho := \int_{\partial S} \frac{1}{z} \neq 0, \quad \textit{even}\ \ \mathrm{im}(\rho) \geq 4\ .$$

Proof. The canonical vertices of the square S are $a := -1 - i$, $b := 1 - i$, $c := 1 + i$ and $d = -1 + i$. Let $p_n = (a_0, a_1, \dots, a_n)$ be an n-equipartition of the segment ab. Because multiplying by i means rotation around the origin in the positive direction (couter-clockwise) by the angle $\pi/2$, $q_n = ip_n := (ia_0, ia_1, \dots, ia_n)$ is an n-equipartition of the segments bc. Similarly, $r_n = iq_n = -p_n$, resp. $s_n = ir_n = -ip_n$, is an n-equipartition of the segment cd, resp. da. Surprisingly, for $f(z) = 1/z$ it holds that

$$C(f,\ p_n) = C(f,\ q_n) = C(f,\ r_n) = C(f,\ s_n)\ .$$

Indeed, extending the fraction by i gives

$$\begin{aligned} C(f,\, p_n) &= \sum_{j=1}^{n} \frac{(b-a)/n}{a + j(b-a)/n} = \sum_{j=1}^{n} \frac{(ib-ia)/n}{ia + j(ib-ia)/n} \\ &= \sum_{j=1}^{n} \frac{(c-b)/n}{b + j(c-b)/n} = C(f,\, q_n) \end{aligned}$$

and similarly for the other two equalities. Furthermore, due to $b - a = 2$ and $a = -1 - i$, extending the fraction by the number $\frac{2j}{n} - 1 + i$ we get that $\mathrm{im}(C(f,\, p_n))$ equals

$$\mathrm{im}\left(\sum_{j=1}^{n} \frac{2/n}{-1-i+2j/n}\right) = \mathrm{im}\left(\frac{2}{n}\sum_{j=1}^{n} \frac{2j/n - 1 + i}{(2j/n-1)^2+1}\right)$$

$$= \frac{2}{n}\sum_{j=1}^{n} \frac{1}{(2j/n-1)^2+1} \geq \frac{2}{n}\sum_{j=1}^{n} \frac{1}{2} = 1\,.$$

By Exercise 7, $\mathrm{im}(\rho) = \mathrm{im}\left(\int_{\partial S} \frac{1}{z}\right)$ is

$$4 \cdot \mathrm{im}\left(\int_{ab} \frac{1}{z}\right) = 4 \cdot \lim_{n\to\infty} \mathrm{im}(C(1/z,\, p_n)) \geq 4 \cdot 1 = 4$$

and indeed $\rho \neq 0$. $\square$

Exercise 7 *Let* (z_n) *be a convergent sequence of complex numbers. Prove that* $\mathrm{im}(\lim\, z_n) = \lim \mathrm{im}(z_n)$.

Exercise 8 **(**$\mathrm{re}(\rho) = 0$**)** *Show that the previous proof gives the equality* $\mathrm{re}(\rho) = 0$.

Exercise 9 ($\rho = 2\pi i$) *Again, let $a := -1 - i$ and $b := 1 - i$. Compute by the Proposition 3 that*

$$\int_{ab} \frac{1}{z} = \frac{\pi i}{2} \ .$$

Thus, by the previous proof,

$$\rho = 4 \cdot \frac{\pi i}{2} = 2\pi i \ .$$

Hint: $\int \frac{1}{1+t^2} = \arctan t$.

Exercise 10 *Let $\varphi(t)\colon [0,1] \to \mathbb{C}$, $\varphi(t) := \mathrm{e}^{2\pi i t}$ and $f(z) := 1/z$. Show that*

$$\int_{\varphi} f = 2\pi i \ .$$

• *Cauchy–Goursat theorem.* This is theorem number 1 in complex analysis: the integral $\int_{\varphi} f$ of any holomorphic function f over the *circuit* φ (i.e., φ is an injective curve except for $\varphi(a) = \varphi(b)$) which lies in the definition domain of f together with with its interior, is 0. We already proved a special case of this theorem in corollary 2. But here we only need to integrate over boundaries of rectangles and need not worry about complicated curves.

Recall that for $X \subset \mathbb{C}$ the *diameter* is defined as

$$\mathrm{diam}(X) = \sup(\{|x - y| \mid x,\, y \in X\}) \ .$$

It may be $+\infty$.

Exercise 11 *If sets A_n with*

$$\mathbb{C} \supset A_1 \supset A_2 \supset \dots$$

are nonempty and closed and let $\lim \mathrm{diam}(A_n) = 0$*, then we have that $\bigcap_{n=1}^{\infty} A_n \neq \emptyset$. Hint: see the proof of Baire's theorem.*

We construct the *quarters* of a rectangle R. Let R have canonical vertices a, b, c, d. When $e := \frac{a+b}{2}$, $f := \frac{b+c}{2}$, $g := \frac{c+d}{2}$ and $h := \frac{d+a}{2}$ are the midpoints of the sides of R and $j := \frac{a+c}{2}$ is its center, then the four *quarters* of R are the rectangles A, B, C and D with the canonical vertices, respectively,

$$(a,\, e,\, j,\, h),\ (e,\, b,\, f,\, j),\ (j,\, f,\, c,\, g)\ \text{ and }\ (h,\, j,\, g,\, d)\ .$$

The rectangle R is divided into quarters by cutting it along the segments eg and hf. For each of these quarters E it clearly holds: $\mathrm{per}(E) = \frac{1}{2}\mathrm{per}(R)$ and $\mathrm{diam}(E) = \frac{1}{2}\mathrm{diam}(R)$.

Theorem 12 (Cauchy–Goursat for rectangles) *Let*

$$f\colon U \to \mathbb{C}$$

be a holomorphic function and $R \subset U$ *be a rectangle. Then*

$$\int_{\partial R} f = 0\ .$$

Proof. Let f, U and R be as shown. We construct nested rectangles

$$R = R_0 \supset R_1 \supset R_2 \supset \dots$$

such that for every $n \in \mathbb{N}_0$, R_{n+1} is a quarter of R_n and

$$\left|\int_{\partial R_{n+1}} f\right| \geq \frac{1}{4}\left|\int_{\partial R_n} f\right| . \tag{1}$$

Let such rectangles $R_0,\ R_1,\ \dots,\ R_n$ be already defined and let A, B, C and D be the quarters of the rectangle R_n. We claim that

$$\int_{\partial R_n} f = \int_{\partial A} f + \int_{\partial B} f + \int_{\partial C} f + \int_{\partial D} f\ . \tag{2}$$

This identity follows from part 3 of the theorem on properties of the integral in the last lecture. After expressing every integral $\int_{\partial A} f, \ldots, \int_{\partial D} f$ as the sum of four integrals over the sides, we get on the right-hand side of the equality (2) 16 term. Eight of them correspond to the sides of quarters inside R_n and mutually cancel because they form four pairs of opposite orientations of the same segment. The remaining eight terms corresponds to the sides of quarters lying on ∂R_n and add up to the integral on the left-hand side of equality (2). Inequalities (1) follow from the triangle inequality: for some quarter $E \in \{A, B, C, D\}$ one has that $|\int_{\partial E} f| \geq \frac{1}{4}|\int_{\partial R_n} f|$. So we set $R_{n+1} = E$.

By Exercise 11 there exists a point z_0 such that

$$z_0 \in \bigcap_{n=0}^{\infty} R_n \ .$$

Since $R_0 = R \subset U$, also $z_0 \in U$. Now we use the existence of the derivative $f'(z_0)$. For a given $\varepsilon > 0$ there exists a $\delta > 0$ such that $B(z_0, \delta) \subset U$ and that, with a function $\Delta \colon B(z_0, \delta) \to \mathbb{C}$, for every $z \in B(z_0, \delta)$ it holds that $|\Delta(z)| < \varepsilon$ (see also Exercise 13) and

$$f(z) = \underbrace{f(z_0) + f'(z_0) \cdot (z - z_0)}_{g(z)} + \underbrace{\Delta(z) \cdot (z - z_0)}_{h(z)} \ .$$

We consider functions $g(z)$ and $h(z)$. It is clear that $g(z)$ is linear and $h(z) = f(z) - g(z)$ is continuous (on $B(z_0, \delta)$). Let $n \in \mathbb{N}_0$ be so large that $R_n \subset B(z_0, \delta)$ (only here we need that $\lim \mathrm{diam}(R_n) = 0$, for the existence point z_0 it is not essential, see Exercise 14). By the linearity of the integral and corollary 2 we have that

$$\int_{\partial R_n} f = \int_{\partial R_n} g + \int_{\partial R_n} h \stackrel{\text{Cor. 2}}{=} \int_{\partial R_n} h \ . \tag{3}$$

Thus

$$\begin{aligned}
\left| \int_{\partial R_n} h \right| &\overset{\text{ML estimate}}{\leq} \max_{z \in \partial R_n} |\Delta(z) \cdot (z - z_0)| \cdot \mathrm{per}(R_n) \\
&< \varepsilon \cdot \mathrm{diam}(R_n) \cdot \mathrm{per}(R_n) \\
&= \varepsilon \cdot \frac{\mathrm{diam}(R)}{2^n} \cdot \frac{\mathrm{per}(R)}{2^n} \\
&< \varepsilon \cdot \frac{\mathrm{per}(R)^2}{4^n}\,. \qquad (4)
\end{aligned}$$

Here we used the above mentioned halving of the diameter and perimeter after quartering and that the diameter of a rectangle is smaller than its perimeter. According to the previous results we have that

$$\frac{1}{4^n}\left| \int_{\partial R} f \right| \overset{\text{ineq. (1)}}{\leq} \left| \int_{\partial R_n} f \right| \overset{\text{eq. (3)}}{=} \left| \int_{\partial R_n} h \right| \overset{\text{ineq. (4)}}{<} \varepsilon \cdot \frac{\mathrm{per}(R)^2}{4^n}$$

$\Rightarrow |\int_{\partial R} f| < \varepsilon \cdot \mathrm{per}(R)^2$. It holds for every $\varepsilon > 0$, so $\int_{\partial R} f = 0$. □

Exercise 13 *What is the value of the function $\Delta(z)$ in the proof at the point z_0?*

Exercise 14 *Prove that for non-emptiness of the intersection in Exercise 11, it suffices to assume that the set A_1 is bounded (instead of the zero limit of diameters). But show also that it does not hold in general metric spaces.*

A remarkable proof! The author of the theorem is the French mathematician *Augustin-Louis Cauchy (1789–1857)*, who also lived a couple of years in Prague during his political exile. However, Cauchy assumed continuity of the derivative f'. It was *Édouard Goursat (1858–1936)* who proved the theorem in 1900 only under the assumption of mere existence of the derivative f':

E. Goursat, Sur la definition générale des fonctions analytiques, d'après Cauchy, *Trans. Amer. Math. Soc.* **1** (1900), 14–16.

The C.–G theorem for rectangles, more precisely their boundaries, will suffice for us, but the theorem holds for general curves. We give just an outline of the proof of the general version.

Theorem 15 (Cauchy–Goursat) *Let $f\colon U \to \mathbb{C}$ be a holomorphic function and $\varphi\colon [a,b] \to U$ be a continuous and piecewise smooth function that is injective, with the exception of the value $\varphi(a) = \varphi(b)$, and whose interior — the bounded component in the pair of components of the set $\mathbb{C} \setminus \varphi[\,[a,b]\,]$ — is a subset of the set U. Then*

$$\int_{\varphi} f = 0\ .$$

Sketch of the proof. We draw in $\mathbb{C}$ with the help of horizontal and vertical lines a fine square grid $\mathcal{M}$. A simple closed curve $\psi\colon [a,b] \to U$ runs along the sides of the grid $\mathcal{M}$ and satisfies that (i) for a given $\varepsilon > 0$ it holds that $|\int_{\varphi} f - \int_{\psi} f| < \varepsilon$ (the curve ψ closely approximates the curve φ) and (ii) the interior of the curve ψ is a subset of the set U. Then

$$\int_{\psi} f = \sum_{R \in M} \int_{\partial R} f = \sum_{R \in M} 0 = 0\ ,$$

where M are those elementary rectangles of the grid $\mathcal{M}$ that lie inside the curve ψ. The first equality holds for the same reason as equality (2) and the first equality in (5) below. The second follows from (ii) and the preceding C.–G. theorems for rectangles. By (i), $|\int_\varphi f| < \varepsilon$. This is true for every ε and $\int_\varphi f = 0$. $\square$

Exercise 16 *Like in Exercise 10, let $\varphi(t)\colon [0,1] \to \mathbb{C}$, $\varphi(t) := \mathrm{e}^{2\pi i t}$ (now we parameterize the whole unit circle) and $f(z) := z^k$, where $k \in \mathbb{Z}$ with $k \neq -1$. Show that*

$$\int_\varphi f = 0\ .$$

(It does not follow completely from the C.–G. theorem!)

• *Independence of $\int$ on the integration rectangle.* We prove that in certain situation the integral $\int_{\partial R} f$ does not depend much on the rectangle R. Recall that every compact set $A \subset \mathbb{C}$ is closed and bounded.

Proposition 17 (independence of $\int_{\partial R} f$ on R) *We assume that $A \subset \mathrm{int}(R) \cap \mathrm{int}(S)$, with a compact set A and rectangles $R, S \subset \mathbb{C}$. Let $f\colon \mathbb{C} \setminus A \to \mathbb{C}$ be a holomorphic function. Then*

$$\int_{\partial R} f = \int_{\partial S} f\ .$$

Proof. Let A, R, S, and f be as given, and let first $S \subset \mathrm{int}(R)$. By extending the sides of the rectangle S, we divide the rectangle R into nine rectangles R_1, R_2, …, R_8, S. Then indeed

$$\int_{\partial R} f \overset{\text{as in (2)}}{=} \sum_{j=1}^{8} \int_{\partial R_j} f + \int_{\partial S} f \overset{\text{Thm. 12, } R_j \subset \mathbb{C}\setminus A}{=} \int_{\partial S} f\ . \qquad (5)$$

We reduce general rectangles R and S to the previous case. By Exercises 18 and 19 for any rectangles R and S and any nonempty compact set $A \subset \mathrm{int}(R) \cap \mathrm{int}(S)$ there is a rectangle T such that

$$A \subset \mathrm{int}(T) \;\text{ and }\; T \subset \mathrm{int}(R) \cap \mathrm{int}(S)\ .$$

By the previous case,

$$\int_{\partial R} f = \int_{\partial T} f = \int_{\partial S} f\ .$$

□

Exercise 18 *Prove that every nonempty intersection of two rectangles is a rectangle.*

Exercise 19 *Prove that for any rectangles R and S and any nonempty compact set $A \subset \mathrm{int}(R) \cap \mathrm{int}(S)$ there exists a rectangle T such that*

$$A \subset \mathrm{int}(T) \;\textit{ and }\; T \subset \mathrm{int}(R) \cap \mathrm{int}(S)\ .$$

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to Mgr. J. Rondoš, Ph.D. by the end of the coming Monday by e-mail (jakub.rondos@gmail.com) solutions to the Exercises 1, 5, 9, 16 and 19.