MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2025/26
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LECTURE 9 (April 20, 2026) INTRODUCTION TO COMPLEX ANALYSIS 1

• *What we prove in the next three lectures.* In this and the next two lectures we prove Theorem 7 below. It states that if a function $f\colon\mathbb{C}\to\mathbb{C}$ has derivative everywhere, then for some complex coefficients a_0, a_1, $\dots$,

$$f(z)=\textstyle\sum_{n\ge 0}a_n z^n\ \ \big(=\lim_{n\to\infty}\sum_{j=0}^{n}a_j z^j\big)$$

for every $z\in\mathbb{C}$.
• *Complex numbers*

$$\mathbb{C}=\{z=a+bi\colon\ a,\,b\in\mathbb{R}\}\ \ (i=\sqrt{-1})$$

form a normed field

$$\mathbb{C}=\langle\mathbb{C},\,0,\,1,\,+,\,\cdot,\,|\cdot|\rangle$$

with the Euclidean norm $|z|=|a+bi|=\sqrt{a^2+b^2}$.

Exercise 1 *Prove the triangle inequality that $|u+v|\le|u|+|v|$ for every numbers $u,v\in\mathbb{C}$.*

Complex numbers form a metric space $\langle\mathbb{C},d\rangle$ with the metric

$$d(z_1,\,z_2)=|z_1-z_2|\,.$$

It is complete and isometric to the Euclidean plane $\mathbb{R}^2$.

Exercise 2 *Prove that $\langle\mathbb{C},d\rangle$ is a complete metric space.*

We denote by U, U_0, U_1, $\dots$ non-empty open subsets of $\mathbb{C}$, and by z the complex variable. Recall the notation

$$\mathrm{re}(a+bi)=a \ \text{ and } \ \mathrm{im}(a+bi)=b$$

for the real and imaginary part of the number $a+bi$. For a number $u\in\mathbb{C}$ and real number $r>0$, we denote by

$$B(u,\, r)=\{z\in\mathbb{C}\colon\ |z-u|<r\}$$

the open disc with the center u and radius $r>0$.

- *Holomorphic functions.* For a function $f\colon U\to\mathbb{C}$ and a point $z_0\in U$, the *derivative* $f'(z_0)$ of f at z_0 is defined as for real functions:

$$f'(z_0)=\lim_{z\to z_0}\tfrac{f(z)-f(z_0)}{z-z_0}\ \ (\in\mathbb{C})\,,$$

if this limit exists. More explicitly, the number $f'(z_0)\in\mathbb{C}$ is the derivative of f at z_0 if and only if for every $\varepsilon>0$ there is a $\delta>0$ such that for every $z\in U$ with $0<|z-z_0|\le\delta$ we have

$$\left|\tfrac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|\le\varepsilon\,.$$

A function $f\colon U\to\mathbb{C}$ is *holomorphic* on U if it has derivative at every point $z_0\in U$. We denote the function

$$U\ni z_0\mapsto f'(z_0)\in\mathbb{C}$$

by f', so that $f'\colon U\to\mathbb{C}$. A function $f\colon\mathbb{C}\to\mathbb{C}$ is *entire* if it is holomorphic on $\mathbb{C}$. The next exercise shows that complex derivatives have the same algebraic properties as in the real case.

Exercise 3 *Prove the next proposition.*

Proposition 4 *Let*

$$f,\, g\colon U \to \mathbb{C} \ \ \textit{and} \ \ h\colon U_0 \to \mathbb{C}$$

be holomorphic functions and $\alpha, \beta \in \mathbb{C}$*. The following hold.*

1. *The function* $\alpha f + \beta g$ *is holomorphic on* U *and* $(\alpha f + \beta g)'$ *equals* $\alpha f' + \beta g'$*.*
2. *The product* fg *is holomorphic on* U *and* $(fg)' = f'g + fg'$*.*
3. *If* $g \neq 0$ *on* U*, then the ratio* f/g *is holomorphic on* U *and* $(f/g)' = (f'g - fg')/g^2$*.*
4. *If* $h[U_0] \subset U$*, then the composite function* $f(h)\colon U_0 \to \mathbb{C}$ *is holomorphic on* U_0 *and* $(f(h))' = f'(h) \cdot h'$*.*

Exercise 5 *Show that* (i) $(z^n)' = nz^{n-1}$ *with* $n \in \mathbb{N}$ *on* $\mathbb{C}$ *and that* (ii) *the derivative of a constant function is the zero function.*

• *Analytic functions.* A function $f\colon U \to \mathbb{C}$ is *analytic* on U if for every point $z_0 \in U$ there exist complex numbers a_0, a_1, $\ldots$ such that for every disc $B = B(z_0, r)$ contained in U and every $z \in B$,

$$f(z) = \textstyle\sum_{n=0}^{\infty} a_n (z - z_0)^n \ \ \big(= \lim_{n\to\infty} \sum_{j=0}^{n} a_j (z_j - z_0)^j\big)\,.$$

Exercise 6 *If* $f\colon U \to \mathbb{C}$ *is analytic then it is holomorphic.*

• *Distinction 1 between complex and real analyses.* In complex analysis the following theorem holds.

Theorem 7 *If* $f\colon \mathbb{C} \to \mathbb{C}$ *is entire, then there exist complex coefficients* a_0, a_1, $\ldots$ *such that for every* $z \in \mathbb{C}$*,*

$$f(z) = \textstyle\sum_{n=0}^{\infty} a_n z^n\,.$$

We will prove this result, but it holds more generally: every function holomorphic on U is analytic on U. For real functions this fails.

Exercise 8 *Let* $f\colon \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = 0$ *for* $x \le 0$*, and by* $f(x) = x^2$ *for* $x \ge 0$*. Then* f *has the following properties.*

1. *Finite derivative* $f'(x)$ *exists for every* $x \in \mathbb{R}$.
2. *The function* f *cannot be expressed on any neighborhood of* 0 *by a power series* $f(x) = \sum_{n\ge 0} a_n x^n$.

The hint for 2 *is that the function expressed by a power series has derivatives of all orders.*

- *Distinction 2 between complex and real analyses.* A function $f\colon U \to \mathbb{C}$ is *bounded* if for some constant $c \ge 0$ we have $|f(z)| \le c$ for every $z \in U$. We will also prove the next theorem.

Theorem 9 (J. Liouville, 1847) *If* $f\colon \mathbb{C} \to \mathbb{C}$ *is entire and bounded, then* f *is constant.*

Again, this does not hold for real functions.

Exercise 10 *Show that the function* $f(x) = \mathrm{e}^{-x^2}\colon \mathbb{R} \to \mathbb{R}$ *is a counterexample to the real Liouville theorem.*

Exercise 11 *Deduce from Liouville's theorem the* FTA *that every non-constant complex polynomial* $p(z)$ *has a root. The hint is to consider the function* $1/p(z)$.

- *Distinction 3 between complex and real analyses.* This distinction concerns the continuity of derivatives.

Corollary 12 *If $f\colon U \to \mathbb{C}$ is a holomorphic function then it has derivatives $f^{(n)}\colon U \to \mathbb{C}$ of all orders $n \in \mathbb{N}$. In particular, $f'\colon U \to \mathbb{C}$ is a continuous function.*

Proof. Holomorphic functions are analytic and analytic functions have derivatives of all orders. $\square$

Exercise 13 *Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that has $f'\colon \mathbb{R} \to \mathbb{R}$ but does not have $f''\colon \mathbb{R} \to \mathbb{R}$.*

Exercise 14 *Describe a function $f\colon \mathbb{R} \to \mathbb{R}$ with discontinuous $f'\colon \mathbb{R} \to \mathbb{R}$.*

• *Distinction 4 between complex and real analyses.* This one is perhaps the most surprising.

Theorem 15 (maximum modulus principle) *Let $f\colon U \to \mathbb{C}$ be holomorphic. Then for every $z_0 \in U$ and every $\delta > 0$ there is $z \in U$ with $0 < |z - z_0| \le \delta$ such that $|f(z)| \ge |f(z_0)|$.*

Thus the modulus function $|f|$ of a holomorphic function f does not have strict local maximum. We will not prove this theorem.

Exercise 16 *The function $f(x) = 1 - x^2$ disproves the maximum modulus principle for real functions.*

• *Segments and their partitions.* In order to eventually prove Theorems 7 and 9, we need integrals over segments and over boundaries of rectangles. We define these geometric objects. For $a, b \in \mathbb{C}$, $a \ne b$, the *segment* $u = ab$ $(\subset \mathbb{C})$ joining the points a and b is the image

$$u = ab = \varphi[\,[0,\,1]\,] = \{\varphi(t)\colon\ 0 \le t \le 1\}\ \ (\subset \mathbb{C})$$

of the interval $[0,1]$ by the linear function

$$\varphi(t) = (b-a)t + a\colon\ [0,\ 1] \to \mathbb{C}$$

with values $\varphi(0) = a$ and $\varphi(1) = a$. The segment is oriented from a to b. So ab and ba are two different segments. The segment ab has *length* $|u| = |ab| = |b-a|$ (≥ 0). A *partition* p of it is a $(k+1)$-tuple $(k \in \mathbb{N})$ $p = \langle a_0, a_1, \ldots, a_k\rangle$ $(\subset u)$ of points

$$a_i = \varphi(t_i),\ \ i = 0,\ 1,\ \ldots,\ k\ ,$$

lying on u that are images of the points t_i in a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0,1]$. So $a_0 = a$, $a_k = b$ and the points $a_0, a_1, \ldots, a_k$ run on u from a to b. The *norm* $\|p\|$ of p is

$$\|p\| = \max_{1\leq i\leq k} |a_{i-1}\, a_i| = \max_{1\leq i\leq k} |a_i - a_{i-1}|\ .$$

Exercise 17 *For every partition* $p = \langle a_0, a_1, \ldots, a_k\rangle$ *of a segment* $u = ab$ *we have* $\sum_{i=1}^k |a_{i-1}a_i| = |ab|$.

• *Cauchy sums.* Let u be a segment, $f\colon\ u \to \mathbb{C}$ be a function, and $p = \langle a_0, a_1, \ldots, a_k\rangle$ be a partition of u. We define the *Cauchy sum* $C(f,p)$, and the *modified Cauchy sum* $C'(f,p)$, by

$$\begin{aligned} C(f,\ p) \ &= \ \textstyle\sum_{i=1}^k f(a_i)\cdot(a_i - a_{i-1})\ \ (\in \mathbb{C})\ \text{ and} \\ C'(f,\ p) \ &= \ \textstyle\sum_{i=1}^k f(a_{i-1})\cdot(a_i - a_{i-1})\ \ (\in \mathbb{C})\,. \end{aligned}$$

Exercise 18 *Show that*

$$|C(f,\ p)|,\ \ |C'(f,\ p)| \leq \sup_{z\in u} |f(z)|\cdot|u|\ \ \ (\in [0,\ +\infty) \cup \{+\infty\})\ .$$

• *Rectangles.* Let $\alpha < \beta$ and $\gamma < \delta$ be real numbers. They determine the *rectangle*

$$R = \{z \in \mathbb{C}\colon\ \alpha \leq \mathrm{re}(z) \leq \beta \wedge \gamma \leq \mathrm{im}(z) \leq \delta\}\ \ (\subset \mathbb{C})\,.$$

The sides of R are parallel to the real and imaginary axes. If $\beta-\alpha = \delta-\gamma$, then R is a *square*. The *canonical vertices* of R is the quadruple $\langle a, b, c, d\rangle \in \mathbb{C}^4$ such that

$$a=\alpha+\gamma i,\ \ b=\beta+\gamma i,\ \ c=\beta+\delta i\ \text{ and }\ d=\alpha+\delta i.$$

It is a counter-clockwise enumeration of the four vertices of R, starting from the bottom left vertex. *The boundary* of the rectangle R is the union of segments

$$\partial R = ab \cup bc \cup cd \cup da\ .$$

The *interior* of R is $\mathrm{int}(R) = R \setminus \partial R$. The *perimeter* of R is the sum of lengths of all four sides,

$$\mathrm{per}(R) = |ab| + |bc| + |cd| + |da|\ .$$

• *Integrals and their existence.* We define the integral $\int_u f$ for a segment u and a continuous function $f\colon u \to \mathbb{C}$. If for every sequence (p_n) of partitions p_n of u with $\lim \|p_n\| = 0$ the limit of corresponding Cauchy sums

$$L = \lim_{n\to\infty} C(f,\, p_n)\ \ (\in \mathbb{C})$$

exists, we say that f has the integral L over u, and write $\int_u f = L$. By the next exercise, this is a correct definition.

Exercise 19 *If the limit L exists for every sequence (p_n) as stated, then L does not depend on (p_n).*

Our definition of $\int_u f$ via Cauchy sums differs from the Riemann integral which uses Riemann sums

$$R(f,\, p,\, q) = \textstyle\sum_{i=1}^{k} f(b_i)(a_i - a_{i-1})\,,$$

where $p = \langle a_0, a_1, \dots, a_k\rangle$ is a partition of the segment $u = ab$ as before and $q = \langle b_1, b_2, \dots, b_k\rangle$ are some tags $b_i \in a_{i-1}a_i$. For continuous functions f there is no difference whether $\int_u f$ is defined by $C(f,p)$ or by $R(f,p,q)$, but in general the two integrals differ. It is well known that for unbounded functions f the Riemann integral $\int_u f$ never exists. Exercise 25 contains an example of an unbounded function f such that the Cauchy integral $\int_u f$ exists. In our lectures we prefer the simple Cauchy integral, but for possibly discontinuous functions the Riemann integral is more satisfactory.

Let R be a rectangle and $f\colon \partial R \to \mathbb{C}$ be a function. We define the integral of f over the boundary of R by the sum

$$\int_{\partial R} f = \int_{ab} f + \int_{bc} f + \int_{cd} f + \int_{da} f\,,$$

if the four integrals over the sides of R exist. Here $\langle a, b, c, d\rangle$ are the canonical vertices of R. We prove the existence theorem for these integrals.

Theorem 20 *Let u be a segment and R be a rectangle.*

1. *If $f\colon u \to \mathbb{C}$ is continuous then the integral $\int_u f$ exists.*
2. *If $f\colon \partial R \to \mathbb{C}$ is continuous then the integral $\int_{\partial R} f$ exists.*

Proof. It suffices to prove part 1. Let $u = ab$ be a segment and let $f\colon u \to \mathbb{C}$ be a continuous function. We show that for every sequence (p_n) of partitions of u with $\lim \|p_n\| = 0$, the sequence $(C(f, p_n))$ of corresponding Cauchy sums is Cauchy. Since $\mathbb{C}$ is a complete metric space, the result follows.

By Exercise 23, it suffices to prove the *Cauchy condition for Cauchy sums* – for every ε there is a δ such that for every two partitions p an q of u with $\|p\|, \|q\| \le \delta$ we have

$$|C(f,\, p) - C(f,\, q)| \le \varepsilon\,.$$

We prove this condition. By Exercise 24 f is uniformly continuous and so for given $\varepsilon > 0$ we take $\delta > 0$ that

$$x,\, y \in u \wedge |x-y| \le \delta \Rightarrow |f(x) - f(y)| \le \tfrac{\varepsilon}{|u|}\,.$$

Let $p = \langle a_0, a_1, \ldots, a_k\rangle$ and $q = \langle b_0, b_1, \ldots, b_l\rangle$ be two partitions of u with $\|p\|, \|q\| \le \delta$. We first suppose that p refines q: $q \subset p$, hence $b_j = a_{i_j}$, $j = 0, 1, \ldots, l$, for some indices $0 = i_0 < i_1 < \cdots < i_l = k$. Then

$$C(f,\, p) \overset{(1)}{=} \textstyle\sum_{j=1}^{l} C(f,\, p_j)\,,$$

where $p_j = \langle a_{i_{j-1}}, a_{i_{j-1}+1}, \ldots, a_{i_j}\rangle$ is the partition of the segment $u_j = a_{i_{j-1}}a_{i_j} = b_{j-1}b_j$, and

$$C(f,\, q) \overset{(2)}{=} \textstyle\sum_{j=1}^{l} C(g_j,\, p_j)\,,$$

where $g_j\colon u_j \to \mathbb{C}$ denotes the function that has the constant value $f(b_j)$ $(= f(a_{i_j}))$ on u_j. Then

$$\begin{aligned}
& |C(f,\, q) - C(f,\, p)| \\
\overset{\text{eqs. (1) and (2), }\Delta\text{-ineq.}}{\le}\; & \textstyle\sum_{j=1}^{l} |C(g_j,\, p_j) - C(f,\, p_j)| \\
\overset{\text{def. of } p_j \text{ and } g_j}{\le}\; & \textstyle\sum_{j=1}^{l} \big|\sum_{m=a_{i_{j-1}+1}}^{a_{i_j}} (f(a_{i_j}) - f(a_m)) \cdot \\
& \cdot (a_m - a_{m-1})\big| \\
\overset{\Delta \text{ ineq., } \delta \text{ and } a_m}{<}\; & \textstyle\sum_{j=1}^{l} \sum_{m=a_{i_{j-1}+1}}^{a_{i_j}} \frac{\varepsilon}{|u|} \cdot |a_m - a_{m-1}| \\
\overset{\text{Exercise 17}}{=}\; & \textstyle\sum_{j=1}^{l} \frac{\varepsilon}{|u|} \cdot |b_j - b_{j-1}| \overset{\text{Exercise 17}}{=} \frac{\varepsilon}{|u|} \cdot |u| = \varepsilon\,.
\end{aligned}$$

Two general partitions are handled by the refinement trick. For given $\varepsilon > 0$ we take the $\delta > 0$ whose existence we proved in the previous paragraph – such that for every two partitions p' and q' of the segment u, where $\|p'\|, \|q'\| \le \delta$ and one of them refines the other, it holds that $|C(f, p') - C(f, q')| \le \frac{\varepsilon}{2}$. Now if p and q are two arbitrary partitions of the segment u with $\|p\|, \|q\| \le \delta$, we take their common refinement, the partition $r = p \cup q$. It refines both p and q and satisfies that $\|r\| \le \delta$. By the definition of δ, we have the desired inequality:

$$\begin{aligned}|C(f, p) - C(f, q)| &\le |C(f, p) - C(f, r)| + \\ &+ |C(f, r) - C(f, q)| \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon\,.\end{aligned}$$

☐

• *Properties of integrals.* We show that integrals are linear, and for continuous f satisfy the ML bound and are additive.

Theorem 21 *Let $u = ab$ be a segment, R be a rectangle and $f, g\colon u, \partial R \to \mathbb{C}$ be two functions, defined on u or on ∂R. The following holds.*

1. *For every $\alpha, \beta \in \mathbb{C}$ the equality*

   $$\textstyle\int_u (\alpha f + \beta g) = \alpha \int_u f + \beta \int_u g$$

   holds if the last two integrals exist. The same linearity holds for the integral $\int_{\partial R}$.

2. *If f is continuous then the integrals $\int_u f$ and $\int_{\partial R} f$ exist and are bounded by the* ML bounds

   $$\textstyle\left|\int_u f\right| \le \max_{z \in u} |f(z)| \cdot |u| \ \text{ and } \ \left|\int_{\partial R} f\right| \le \max_{z \in \partial R} |f(z)| \cdot \mathrm{per}(R)\,.$$

3. *If* $f\colon u\to\mathbb{C}$ *is continuous then* $\int_{ba} f = -\int_{ab} f$.

4. *Let* c *be an interior point of* $u = ab$*, so that* $c \in ab$ *and* $c \neq a, b$*. If* $f\colon u\to\mathbb{C}$ *is continuous, then* $\int_{ab} f = \int_{ac} f + \int_{cb} f$.

Proof. 1. Let $\alpha, \beta \in \mathbb{C}$ and $f, g\colon u \to \mathbb{C}$ be such that the integrals $\int_u f$ and $\int_u g$ exist. Let (p_n) be any sequence of partitions of u with $\lim \|p_n\| = 0$. Then

$$\begin{aligned}\lim_{n\to\infty} C(\alpha f + \beta g,\, p_n) &= \lim\big(\alpha C(f,\, p_n) + \beta C(g,\, p_n)\big)\\ &= \alpha \lim C(f,\, p_n) + \beta \lim C(g,\, p_n)\\ &= \alpha \int_u f + \beta \int_u g\,,\end{aligned}$$

which proves the former linearity. The latter linearity follows from the former.

2. The maxima exist by Exercise 22. The former bound follows by a limit transition from the definition of $\int_u f$ and from Exercise 18. The latter bound follows from the former.

3. Now we use the modified Cauchy sums $C'(f, p)$. Since f is uniformly continuous (Exercise 24),

$$C(f,\, p_n) = C'(f,\, p_n) + o(1) \quad (n \to +\infty)$$

for every sequence (p_n) of partitions of u with $\lim \|p_n\| = 0$. But then $\lim C(f, p_n) = \int_{ab} f$ and $\lim C'(f, p_n) = \int_{ba} (-f)$.

4. Let c be an inner point of ab and let $f\colon u \to \mathbb{C}$ be a continuous function. Let (p_n) be a sequence of partitions of ab such that $\lim \|p_n\| = 0$. The point c splits in the obvious way every p_n in a partition q_n of ac and a partition r_n of cb; if c is inside a subsegment of p_n, we split the subsegment in two. Clearly, $\|q_n\|, \|r_n\| \le \|p_n\|$. Since f is uniformly continuous (Exercise 24),

$$C(f,\, p_n) = C(f,\, q_n) + C(f,\, r_n) + o(1) \quad (n \to +\infty)\,.$$

The identity $\int_{ab} f = \int_{ac} f + \int_{cb} f$ follows by limit transition. $\square$

Exercise 22 *Why do the two maxima in part 2 exist?*

Exercise 23 *Let u be a segment and*

$$f\colon u \to \mathbb{C}$$

be a continuous function. Show that if the Cauchy condition for Cauchy sums holds, then for every sequence (p_n) of partitions of u with $\lim \|p_n\| = 0$ the sequence of Cauchy sums $(C(f, p_n))$ $(\subset \mathbb{C})$ is Cauchy.

Exercise 24 *Let $A \subset M$ be a compact set in a metric space $\langle M, d\rangle$ and let $f\colon A \to N$ be a continuous function to the metric space $\langle N, e\rangle$. Prove that then f is uniformly continuous,*

$$\forall\,\varepsilon\;\exists\,\delta\,\big(a,\,b \in A \wedge d(a,\,b) \le \delta \Rightarrow e(f(a),\,f(b)) \le \varepsilon\big)\,.$$

Exercise 25 *Let $f\colon [0,1] \to [0,+\infty)$ be given by $f(x) = \frac{1}{\sqrt{x}}$ for $x > 0$ and $f(0) = 0$. We regard the interval $[0,1]$ as the complex segment $u = 0\,1$. Although the function f is unbounded, show that the (Cauchy) integral $\int_u f$ exists, and compute it.*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 2, 8, 11, 17 a 24.