MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2024/25
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LECTURE 9 (April 16, 2025) INTRODUCTION TO COMPLEX ANALYSIS 1 (revised on April 22)

• *What we prove in the next three lectures.* In this and the next two lectures we prove Theorem 7 stated below. It says that if a function $f\colon \mathbb{C} \to \mathbb{C}$ has derivative everywhere, then for some coefficients $a_n \in \mathbb{C}$, $n = 0, 1, \dots$, we have for every $z \in \mathbb{C}$ that

$$f(z) = \textstyle\sum_{n\ge 0} a_n z^n \ \ \left(= \lim_{n\to\infty} \sum_{j=0}^{n} a_j z^j\right).$$

• *Complex numbers*

$$\mathbb{C} = \{z = a + bi\colon\ a,\, b \in \mathbb{R}\}\ \ (i = \sqrt{-1})$$

form a normed field

$$\mathbb{C}_{\mathrm{OF}} = (\mathbb{C},\, 0,\, 1,\, +,\, \cdot,\, |\cdot|)\,.$$

The norm is Euclidean one, $|z| = |a + bi| = \sqrt{a^2 + b^2}$.

Exercise 1 *Prove the triangle inequality that for every numbers $u, v \in \mathbb{C}$ we have $|u + v| \le |u| + |v|$.*

Complex numbers form a metric space $(\mathbb{C}, d)$ with the metric

$$d(z_1,\, z_2) = |z_1 - z_2|\,.$$

It is complete and is isometric to the Euclidean plane $\mathbb{R}^2$.

Exercise 2 *Prove that $(\mathbb{C}, d)$ is a complete metric space.*

Non-empty open subsets of $\mathbb{C}$ are denoted by $U, U_0, U_1, \ldots$, and z is the complex variable. Recall the notation

$$\mathrm{re}(a+bi) = a \ \text{ and } \ \mathrm{im}(a+bi) = b$$

for the real and imaginary part of the number $a+bi$. For a given $u \in \mathbb{C}$ and $r > 0$, we denote by

$$B(u,\, r) = \{z \in \mathbb{C}\colon\ |z-u| < r\}$$

the open disc with the center u and radius $r > 0$.

• *Holomorphic functions.* For a function $f\colon U \to \mathbb{C}$ and a point $z_0 \in U$, the *derivative* $f'(z_0)$ of f at z_0 is defined as for real functions:

$$f'(z_0) = \lim_{z\to z_0} \tfrac{f(z)-f(z_0)}{z-z_0} \ \ (\in \mathbb{C})\,,$$

if this limit exists. More explicitly, the number $f'(z_0) \in \mathbb{C}$ is the derivative of f at z_0 if and only if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for every $z \in U$ with $0 < |z-z_0| \le \delta$ we have

$$\left|\tfrac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| \le \varepsilon\,.$$

We call a function $f\colon U \to \mathbb{C}$ *holomorphic* on U if it has derivative at every point $z_0 \in U$. We denote the function

$$U \ni z_0 \mapsto f'(z_0) \in \mathbb{C}$$

by f', so that $f'\colon U \to \mathbb{C}$. A function $f\colon \mathbb{C} \to \mathbb{C}$ is called *entire* if it is holomorphic on $\mathbb{C}$. The next exercise shows that complex derivatives have the same algebraic properties as real ones.

Exercise 3 *Prove the next proposition.*

Proposition 4 (properties of derivatives) *Let*

$$f,\, g\colon U \to \mathbb{C} \ \ and \ \ h\colon U_0 \to \mathbb{C}$$

be holomorphic functions and $\alpha, \beta \in \mathbb{C}$*. The following hold.*

1. *The function* $\alpha f + \beta g$ *is holomorphic on* U *and* $(\alpha f + \beta g)'$ *equals* $\alpha f' + \beta g'$*.*

2. *The product* fg *is holomorphic on* U *and* $(fg)' = f'g + fg'$*.*

3. *If* $g \neq 0$ *on* U*, then the ratio* f/g *is holomorphic on* U *and* $(f/g)' = (f'g - fg')/g^2$*.*

4. *If* $h[U_0] \subset U$*, then the composite function* $f(h)\colon U_0 \to \mathbb{C}$ *is holomorphic on* U_0 *and* $(f(h))' = f'(h) \cdot h'$*.*

Exercise 5 *Show that* (i) *(*$n \in \mathbb{N}$*)* $(z^n)' = nz^{n-1}$ *on* $\mathbb{C}$ *and* (ii) *the derivative of a constant function is the zero function.*

• *Analytic functions.* The function $f\colon U \to \mathbb{C}$ is *analytic* on U if for every point $z_0 \in U$ there exist numbers a_n in $\mathbb{C}$, $n = 0, 1, \dots,$ such that for every open disc $B = B(z_0, r)$ contained in U we have for every $z \in B$ that

$$f(z) = \textstyle\sum_{n=0}^{\infty} a_n(z - z_0)^n \ \ \left(= \lim_{n\to\infty} \sum_{j=0}^{n} a_j(z_j - z_0)^j\right).$$

Exercise 6 *If* $f\colon U \to \mathbb{C}$ *is analytic then it is holomorphic.*

• *The first difference of analysis in* $\mathbb{C}$ *and analysis in* $\mathbb{R}$*.* In complex analysis the following theorem holds.

Theorem 7 (holomorphic ⇒ analytic) *If* $f\colon \mathbb{C} \to \mathbb{C}$ *is an entire function, then there exist coefficients* a_n *in* $\mathbb{C}$*,* $n = 0,\ 1,\ \dots,$ *such that for every number* $z \in \mathbb{C}$ *we have*

$$f(z) = \textstyle\sum_{n=0}^{\infty} a_n z^n\,.$$

In our three lectures we prove only this result for entire functions. It holds more generally that every holomorphic function on U is analytic on U. For real functions this is not true.

Exercise 8 *We define a function*

$$f\colon \mathbb{R} \to \mathbb{R}$$

by $f(x) = 0$ for $x \le 0$ and by $f(x) = x^2$ for $x \ge 0$. Prove that (i) *f has finite $f'(x)$ ($\in \mathbb{R}$) for every $x \in \mathbb{R}$ but* (ii) *f cannot be expressed on any neighborhood of 0 by a power series $f(x) = \sum_{n\ge 0} a_n x^n$. The hint for* (ii) *is that the function expressed by a power series has derivatives of all orders.*

• *The second difference of analysis in $\mathbb{C}$ and analysis in $\mathbb{R}$.* A function $f\colon U \to \mathbb{C}$ is *bounded* if for some constant $c \ge 0$ we have $|f(z)| \le c$ for every $z \in U$. In our three lectures we prove also the following theorem.

Theorem 9 (J. Liouville, 1847) *If $f\colon \mathbb{C} \to \mathbb{C}$ is entire and bounded, then f is constant.*

This again is not true for real functions:

Exercise 10 *Show that the function $f(x) = \mathrm{e}^{-x^2}\colon \mathbb{R} \to \mathbb{R}$ is a counterexample to the real Liouville theorem.*

Exercise 11 *Deduce from Liouville's theorem the Fundamental Theorem of Algebra that every non-constant polynomial $p(z)$ in $\mathbb{C}[z]$ has a root. The hint is to consider the function $1/p(z)$.*

• *The third difference of analysis in $\mathbb{C}$ and analysis in $\mathbb{R}$* concerns the continuity of derivatives.

Corollary 12 (all derivatives) *If $f\colon U \to \mathbb{C}$ is a holomorphic function then it has derivatives $f^{(n)}\colon U \to \mathbb{C}$ of all orders $n \in \mathbb{N}$. In particular, $f'\colon U \to \mathbb{C}$ is a continuous function.*

Proof. Holomorphic functions are analytic and analytic functions have derivatives of all orders. □

Exercise 13 *Find a function $f\colon \mathbb{R} \to \mathbb{R}$ that has $f'\colon \mathbb{R} \to \mathbb{R}$ but does not have $f''\colon \mathbb{R} \to \mathbb{R}$.*

Exercise 14 *Describe a function $f\colon \mathbb{R} \to \mathbb{R}$ with discontinuous $f'\colon \mathbb{R} \to \mathbb{R}$.*

• *The fourth difference of analysis in $\mathbb{C}$ and analysis in $\mathbb{R}$* is perhaps the most surprising one.

Theorem 15 (maximum modulus principle) *Let $f\colon U \to \mathbb{C}$ be a holomorphic function. Then for every point $z_0 \in U$ and every $\delta > 0$ there is a point $z \in U$ with $0 < |z - z_0| \le \delta$ such that $|f(z)| \ge |f(z_0)|$.*

Thus the modulus function $|f|$ of a holomorphic function f does not have strict local maximum. We will not prove this theorem.

Exercise 16 *The function $f(x) = 1 - x^2$ disproves the maximum modulus principle for real functions.*

• *Segments and their partitions.* In order to prove Theorems 7 and 9 we need integrals over segments and over boundaries of rectangles. We define these geometric objects. For $a, b \in \mathbb{C}$, $a \ne b$, the *segment* $u = ab$ $(\subset \mathbb{C})$ spanned by the points a and b is the image

$$u = ab = \varphi[\,[0,\, 1]\,] = \{\varphi(t)\colon\ 0 \le t \le 1\}\ \ (\subset \mathbb{C})$$

of the interval $[0, 1]$ by the linear function

$$\varphi(t) = (b - a)t + a\colon [0,\, 1] \to \mathbb{C}$$

which has values $\varphi(0) = a$ and $\varphi(1) = a$. The segment is oriented from a to b. So ab and ba are two different segments. The segment ab has *length* $|u| = |ab| = |b - a|$ (≥ 0). A *partition* p of the segment $u = ab$ is a $(k + 1)$-tuple, $k \in \mathbb{N}$, $p = (a_0, a_1, \ldots, a_k)$ $(\subset u)$ of the points

$$a_i = \varphi(t_i), \quad i = 0,\, 1,\, \ldots,\, k\ ,$$

lying on u, which are images of the points t_i in a partition $0 = t_0 < t_1 < \cdots < t_k = 1$ of the interval $[0, 1]$. So $a_0 = a$, $a_k = b$ and the points $a_0, a_1, \ldots, a_k$ run on u from a to b. The *norm* $\|p\|$ of p is

$$\|p\| = \max_{1 \leq i \leq k} |a_{i-1}a_i| = \max_{1 \leq i \leq k} |a_i - a_{i-1}|\ .$$

Exercise 17 *For every partition $p = (a_0, a_1, \ldots, a_k)$ of a segment $u = ab$ we have $\sum_{i=1}^{k} |a_{i-1}a_i| = |ab|$.*

• *Cauchy sums.* Let u be a segment, $f\colon u \to \mathbb{C}$ be a function and $p = (a_0, a_1, \ldots, a_k)$ be a partition of u. We define the *Cauchy sum* $C(f, p)$ and the *modified Cauchy sum* $C'(f, p)$ by

$$\begin{aligned} C(f,\, p) &= \textstyle\sum_{i=1}^{k} f(a_i) \cdot (a_i - a_{i-1}) \ \ (\in \mathbb{C}) \ \text{ and} \\ C'(f,\, p) &= \textstyle\sum_{i=1}^{k} f(a_{i-1}) \cdot (a_i - a_{i-1}) \ \ (\in \mathbb{C})\,. \end{aligned}$$

This terminology is justified by the fact that A.-L. Cauchy considered these sums already in 1823 when he defined complex integrals for continuous functions.

Exercise 18 *Show that*

$$|C(f,\, p)|,\ |C'(f,\, p)| \leq \sup_{z \in u} |f(z)| \cdot |u| \quad (\in [0,\, +\infty) \cup \{+\infty\})\ .$$

• *Rectangles.* Let $\alpha < \beta$ and $\gamma < \delta$ be four real numbers. They determine the *rectangle*

$$R = \{z \in \mathbb{C}\colon\ \alpha \le \mathrm{re}(z) \le \beta \wedge \gamma \le \mathrm{im}(z) \le \delta\}\ \ (\subset \mathbb{C})\,.$$

R has sides parallel to the real and imaginary axis. If $\beta - \alpha = \delta - \gamma$, we call R a *square*. The *canonical vertices* of R is the quadruple (a, b, c, d) in $\mathbb{C}^4$ such that

$$a = \alpha + \gamma i,\ \ b = \beta + \gamma i,\ \ c = \beta + \delta i\ \text{ and }\ d = \alpha + \delta i.$$

It is a counter-clockwise enumeration of the four vertices of R, starting from the bottom left vertex. *The boundary* of the rectangle R is the union of segments

$$\partial R = ab \cup bc \cup cd \cup da\ .$$

The *interior* of R is $\mathrm{int}(R) = R \setminus \partial R$. The *perimeter* of R is the sum of lengths of all four sides,

$$\mathrm{per}(R) = |ab| + |bc| + |cd| + |da|\ .$$

• *Integrals and their existence.* We define the integral $\int_u f$ for a segment u and a (usually continuous) function $f\colon u \to \mathbb{C}$. If for every sequence (p_n) of partitions p_n of u with $\lim \|p_n\| = 0$ the limit of corresponding Cauchy sums

$$L = \lim_{n\to\infty} C(f,\, p_n)\ \ (\in \mathbb{C})$$

exists, we say that f has the integral L over u and set $\int_u f = L$.

Exercise 19 *Prove that if the limit L exists for every sequence (p_n) as stated, then L does not depend on (p_n). Thus $\int_u f$ is correctly defined. Hint – interlace two sequences (p_n) and (q_n) as $(p_1, q_1, p_2, q_2, \dots)$.*

Our definition of $\int_u f$ via Cauchy sums differs from the Riemann integral which uses Riemann sums

$$R(f,\, p,\, q) = \sum_{i=1}^{k} f(b_i)(a_i - a_{i-1})\,,$$

where $p = (a_0, a_1, \ldots, a_k)$ is a partition of the segment $u = ab$ as before and $q = (b_1, b_2, \ldots, b_k)$ are some tags $b_i \in a_{i-1}a_i$. For continuous functions f there is no difference whether $\int_u f$ is defined by $C(f, p)$ or by $R(f, p, q)$, but in general the two integrals differ. It is well known that for unbounded functions f the Riemann integral $\int_u f$ never exists. Exercise 25 contains an example of an unbounded function f such that the Cauchy integral $\int_u f$ exists. In our lectures we prefer the simple Cauchy integral, but for possibly discontinuous functions the Riemann integral is more satisfactory.

Let R be a rectangle and $f\colon \partial R \to \mathbb{C}$ be a function. We define the integral of f over the boundary of R by the sum

$$\int_{\partial R} f = \int_{ab} f + \int_{bc} f + \int_{cd} f + \int_{da} f\,,$$

if these four integrals over the sides of R exist. Here (a, b, c, d) are the canonical vertices of R. We obtain the basic existence theorem for these integrals.

Theorem 20 (existence of $\int$) *Suppose that u is a segment and R is a rectangle.*

1. *If $f\colon u \to \mathbb{C}$ is continuous then the integral $\int_u f$ exists.*
2. *If $f\colon \partial R \to \mathbb{C}$ is continuous then the integral $\int_{\partial R} f$ exists.*

Proof. Clearly, part 2 follows from part 1, which we prove. Let $u = ab$ be a segment and $f\colon u \to \mathbb{C}$ be a continuous function. We show that for every sequence (p_n) of partitions of u with $\lim \|p_n\| =$

0 the sequence $(C(f, p_n))$ of corresponding Cauchy sums is Cauchy. Since $\mathbb{C}$ is a complete metric space, the result follows.

By Exercise 23, it suffices to prove the *Cauchy condition for Cauchy sums* – for every ε there is a δ such that for every two partitions p an q of u with $\|p\|, \|q\| \le \delta$ we have (f is continuous)

$$|C(f,\, p) - C(f,\, q)| \le \varepsilon\,.$$

We prove this Cauchy condition. By Exercise 24 f is uniformly continuous and so for the given $\varepsilon > 0$ we take a $\delta > 0$ that

$$x,\, y \in u \wedge |x - y| \le \delta \Rightarrow |f(x) - f(y)| \le \tfrac{\varepsilon}{|u|}\,.$$

Let $p = (a_0, a_1, \dots, a_k)$ and $q = (b_0, b_1, \dots, b_l)$ be two partitions of u with $\|p\|, \|q\| \le \delta$. First suppose that p refines q: $q \subset p$, hence $b_j = a_{i_j}$, $j = 0, 1, \dots, l$, for some indices $0 = i_0 < i_1 < \cdots < i_l = k$. Then

$$C(f,\, p) \overset{(1)}{=} \textstyle\sum_{j=1}^{l} C(f,\, p_j)\,,$$

where $p_j = (a_{i_{j-1}}, a_{i_{j-1}+1}, \dots, a_{i_j})$ is the partition of the segment $u_j = a_{i_{j-1}}a_{i_j} = b_{j-1}b_j$, and

$$C(f,\, q) \overset{(2)}{=} \textstyle\sum_{j=1}^{l} C(g_j,\, p_j)\,,$$

where $g_j\colon u_j \to \mathbb{C}$ denotes the function that has the constant value

$f(b_j)$ $(= f(a_{i_j}))$ on u_j. Then

$$\begin{aligned}
& |C(f,\,q) - C(f,\,p)| \\
\stackrel{\text{eqs. (1) and (2), }\Delta\text{-ineq.}}{\le} \; & \textstyle\sum_{j=1}^{l} |C(g_j,\,p_j) - C(f,\,p_j)| \\
\stackrel{\text{def. of } p_j \text{ and } g_j}{\le} \; & \textstyle\sum_{j=1}^{l} \big| \sum_{m=a_{i_{j-1}+1}}^{a_{i_j}} (f(a_{i_j}) - f(a_m)) \cdot \\
& \cdot (a_m - a_{m-1}) \big| \\
\stackrel{\Delta \text{ ineq., } \delta \text{ and } a_m}{<} \; & \textstyle\sum_{j=1}^{l} \sum_{m=a_{i_{j-1}+1}}^{a_{i_j}} \frac{\varepsilon}{|u|} \cdot |a_m - a_{m-1}| \\
\stackrel{\text{Exercise 17}}{=} \; & \textstyle\sum_{j=1}^{l} \frac{\varepsilon}{|u|} \cdot |b_j - b_{j-1}| \stackrel{\text{Exercise 17}}{=} \frac{\varepsilon}{|u|} \cdot |u| = \varepsilon\,.
\end{aligned}$$

For two general partitions we use the refinement trick. For a given $\varepsilon > 0$ we take the $\delta > 0$ whose existence we proved in the previous paragraph, i.e., such that for every two partitions p' and q' of the segment u, where $\|p'\|, \|q'\| \le \delta$ and one of them refines the other, it holds that $|C(f,\,p') - C(f, q')| \le \frac{\varepsilon}{2}$. Now if p and q are two arbitrary partitions of the segment u with $\|p\|, \|q\| \le \delta$, we take their common refinement, the partition $r = p \cup q$. It refines both p and q and satisfies that $\|r\| \le \delta$. By the definition of δ, we have the desired inequality:

$$\begin{aligned}
|C(f,\,p) - C(f,\,q)| \;&\le\; |C(f,\,p) - C(f,\,r)| + \\
&+\; |C(f,r) - C(f,q)| \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon\,.
\end{aligned}$$

□

• *Properties of integrals.* We show that integrals are linear, and for continuous f satisfy the ML bound and are additive.

Theorem 21 (properties of $\int$) *Suppose that $u = ab$ is a segment, R is a rectangle and $f, g\colon u, \partial R \to \mathbb{C}$ are two functions, both defined on u or on ∂R. The following holds.*

1. *For every* $\alpha, \beta \in \mathbb{C}$ *the equality*

$$\int_u (\alpha f + \beta g) = \alpha \int_u f + \beta \int_u g$$

holds if the last two integrals exist. The same linearity holds for the integral $\int_{\partial R}$.

2. *If* f *is continuous then the integrals* $\int_u f$ *and* $\int_{\partial R} f$ *exist and are bounded by the* ML bounds

$$\left| \int_u f \right| \le \max_{z \in u} |f(z)| \cdot |u| \ \ \textit{and} \ \ \left| \int_{\partial R} f \right| \le \max_{z \in \partial R} |f(z)| \cdot \mathrm{per}(R)\,.$$

3. *If* $f\colon u \to \mathbb{C}$ *is continuous then* $\int_{ba} f = -\int_{ab} f$.

4. *Let* c *be an interior point of* $u = ab$, *which means that* $c \in ab$ *and* $c \ne a, b$. *If* $f\colon u \to \mathbb{C}$ *is continuous then* $\int_{ab} f = \int_{ac} f + \int_{cb} f$.

Proof. 1. Let $\alpha, \beta \in \mathbb{C}$ and $f, g\colon u \to \mathbb{C}$ be such that the integrals $\int_u f$ and $\int_u g$ exist. Let (p_n) be any sequence of partitions of u with $\lim \|p_n\| = 0$. Then

$$\begin{aligned} \lim C(\alpha f + \beta g,\, p_n) &= \lim \big(\alpha C(f,\, p_n) + \beta C(g,\, p_n)\big) \\ &= \alpha \lim C(f,\, p_n) + \beta \lim C(g,\, p_n) \\ &= \alpha \int_u f + \beta \int_u g\,, \end{aligned}$$

which proves the former linearity. The latter linearity follows from the former.

2. The maxima exist by Exercise 22. The former bound follows by a limit transition from the definition of $\int_u f$ and from Exercise 18. The latter bound follows from the former.

3. Now we use the modified Cauchy sums $C'(f, p)$. Since f is uniformly continuous (Exercise 24),

$$C(f,\, p_n) = C'(f,\, p_n) + o(1) \ \ (n \to +\infty)$$

for every sequence (p_n) of partitions of u with $\lim \|p_n\| = 0$. But then $\lim C(f, p_n) = \int_{ab} f$ and $\lim C'(f, p_n) = \int_{ba}(-f)$.

4. Let c be an inner point of ab and let $f\colon u \to \mathbb{C}$ be a continuous function. Let (p_n) be a sequence of partitions of ab such that $\lim \|p_n\| = 0$. The point c splits in the obvious way every p_n in a partition q_n of ac and a partition r_n of cb; if c is inside a subsegment of p_n, we split the subsegment in two. Clearly, $\|q_n\|, \|r_n\| \le \|p_n\|$. Since f is uniformly continuous (Exercise 24),

$$C(f,\, p_n) = C(f,\, q_n) + C(f,\, r_n) + o(1) \ \ (n \to +\infty)\,.$$

The identity $\int_{ab} f = \int_{ac} f + \int_{cb} f$ follows by limit transition. □

Exercise 22 *Explain why the two maxima in part 2 of the theorem exist.*

Exercise 23 *Let u be a segment and*

$$f\colon u \to \mathbb{C}$$

be a continuous function. Show that if the Cauchy condition for Cauchy sums holds, then for every sequence (p_n) of partitions of u with $\lim \|p_n\| = 0$ the sequence of Cauchy sums $(C(f, p_n))$ $(\subset \mathbb{C})$ is Cauchy.

Exercise 24 *Let $A \subset M$ be a compact set in a metric space (M, d) and let $f\colon A \to N$ be a continuous function to the metric space (N, e). Prove that then f is uniformly continuous, that is,*

$$\forall\, \varepsilon\ \exists\, \delta\, \big(a,\, b \in A \wedge d(a,\, b) \le \delta \Rightarrow e(f(a),\, f(b)) \le \varepsilon\big)\,.$$

Exercise 25 *Let* $f\colon [0,1] \to [0,+\infty)$ *be given by* $f(x) = \frac{1}{\sqrt{x}}$ *for* $x > 0$ *and* $f(0) = 0$*. We regard the interval* $[0,1]$ *as the complex segment* $u = 01$*. Although the function* f *is unbounded, show that the (Cauchy) integral* $\int_u f$ *exists and compute it.*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 2, 8, 11, 17 a 24.