MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2025/26
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LECTURE 8 (April 13, 2026) PÓLYA'S THEOREM (1921) ON RANDOM WALKS IN $\mathbb{Z}^d$ VIA POWER SERIES

• *Graphs and walks.* Before we state the theorem, which can be classified equally well to belong to probability theory or (as we approach it here) to enumerative combinatorics, we need some notions. A *graph* $G = \langle V, E \rangle$ consists of the set of *vertices* V and the set of *edges* $E \subset \binom{V}{2}$. Here

$$\binom{V}{2} = \{A\colon\ A \subset V \wedge |A| = 2\}$$

is the set of all two-element subsets of V.

Exercise 1 *Find a formula for the number of all graphs with an n-element vertex set V.*

In Pólya's theorem, we are interested in certain infinite graphs. $G = \langle V, E \rangle$ is *locally finite* (LF) if for every vertex $v \in V$ there are only finitely many edges $e \in E$ such that $v \in e$. If for every $v \in V$ there are exactly d such edges, we call G *d-regular*. A *walk* w in G is a finite sequence

$$w = \langle v_0,\, v_1,\, \dots,\, v_n \rangle$$

of vertices $v_i \in V$ such that $\{v_{i-1}, v_i\} \in E$ for every $i = 1, 2, \dots, n$. We denote the *length* n of w by $|w|$. We call v_0 the *start of the walk* w. For any LF graph G and its vertex v, let $d_n(v, G)$ be the number of walks in G with start v and length n.

Exercise 2 *Let G be a d-regular graph and v be its vertex. Then $d_n(v, G) = d^n$.*

A *recurrent walk* $\langle v_0, v_1, \ldots, v_n \rangle$ in a graph G revisits the start: there exists $i \in \{1, 2, \ldots, n\}$ such that $v_i = v_0$. Let $a_n(v, G)$ be the number of recurrent walks in G starting at v and with length n. An *automorphism* of $G = \langle V, E \rangle$ is a bijection $f\colon V \to V$ such that $f[e], f^{-1}[e] \in E$ for every $e \in E$.

Exercise 3 *Describe all automorphisms of the path P_6 and the circle C_6 of length 6. Here $V = \{1, 2, \ldots, 6\}$, P_6 has edges*

$$E = \{\{1,2\},\ \{2,3\},\ \{3,4\},\ \{4,5\},\ \{5,6\}\}$$

and C_6 has edges

$$E = \{\{1,2\},\ \{2,3\},\ \{3,4\},\ \{4,5\},\ \{5,6\},\ \{6,1\}\}\ .$$

$G = \langle V, E \rangle$ is *transitive* if, for every two vertices $u, v \in V$, there exists an automorphism f of G such that $f(u) = v$.

Exercise 4 *Prove the following proposition.*

Proposition 5 *Let $G = \langle V, E \rangle$ be a transitive and locally finite graph. Then for every length $n \in \mathbb{N}_0$ and every two vertices $u, v \in V$, we have $d_n(u, G) = d_n(v, G)$ and $a_n(u, G) = a_n(v, G)$.*

For any transitive and LF graph G, we therefore denote the number of walks and recurrent walks with length n, starting at a fixed vertex, by $d_n(G)$ and $a_n(G)$, respectively.

Exercise 6 *Show by examples that in a general graph the numbers of walks with a given length depend on the starting vertex.*

Exercise 7 *Prove that the infinite path*

$$P = \langle \mathbb{Z},\, \{\{n,\, n+1\}\colon\ n \in \mathbb{Z}\}\rangle$$

is a transitive graph. How many recurrent walks with length 5 *(and a fixed starting vertex) are there in* P*?*

We consider more general graphs ($d \in \mathbb{N}$)

$$G_d = \langle \mathbb{Z}^d,\, \{\{\overline{u},\, \overline{v}\}\colon\ \textstyle\sum_{i=1}^{d} |u_i - v_i| = 1\}\rangle\,,$$

where we write $\overline{u} = (u_1, \ldots, u_d) \in \mathbb{Z}^d$.

Exercise 8 *The graphs* G_d *are transitive and* G_d *is* $2d$*-regular.*

• *Pólya's theorem.* Here is the celebrated theorem of Pólya.

Theorem 9 (G. Pólya, 1921) *For* $d = 1$ *and* 2 *we have*

$$\lim_{n\to\infty} \frac{a_n(\mathbb{Z}^d)}{d_n(\mathbb{Z}^d)} = \lim_{n\to\infty} \frac{a_n(\mathbb{Z}^d)}{(2d)^n} = 1\,.$$

For $d \ge 3$ *we have*

$$\lim_{n\to\infty} \frac{a_n(\mathbb{Z}^d)}{d_n(\mathbb{Z}^d)} = \lim_{n\to\infty} \frac{a_n(\mathbb{Z}^d)}{(2d)^n} < 1\,.$$

In other words, in dimensions $d \le 2$ for large n, a random walk of length n almost certainly revisits the start. In dimensions $d \ge 3$, it never revisits the start with probability > 0.

• *Weak Abel's theorem.* We make use of the following theorem about power series.

Theorem 10 (weak Abel's) *Let* $R > 0$ *be a real number and* $\sum_{n=0}^{\infty} u_n x^n \in \mathbb{R}[[x]]$ *be a power series with all coefficients* $u_n \ge 0$

that converges for every $x \in [0, R)$. Then the next functional limit and sum of an infinite series are always defined and equal,

$$\lim_{x \to R} \sum_{n=0}^{\infty} u_n x^n = \lim_{n \to \infty} \sum_{j=0}^{n} u_j R^j \quad (\in [0, +\infty) \cup \{+\infty\}) .$$

They may be both equal to $+\infty$.

Proof. For every $N \in \mathbb{N}$ we have

$$\begin{aligned} \sum_{n=0}^{N} u_n R^n &= \lim_{x \to R^-} \sum_{n=0}^{N} u_n x^n \\ &\leq \lim_{x \to R} \sum_{n=0}^{\infty} u_n x^n \leq \sum_{n=0}^{\infty} u_n R^n . \end{aligned}$$

All limits and sums are defined, possibly with the value $+\infty$, due to monotonicity and nonnegativity. The first equality follows from the continuity of polynomials. The next two inequalities follow from the nonnegativity of u_n and from $x \in (0, R)$. The limit transition $N \to +\infty$ gives the claim. □

Exercise 11 *Explain why in this proof we write first the left-sided limit $x \to R^-$, but then the two-sided limit $x \to R$.*

• *Proof of Pólya's theorem by power series.* We prove Pólya's Theorem 9 only in dimensions $d = 2$ and 3. The symbols d_n and a_n denote, respectively, the number of walks and of recurrent walks in the graph G_d with length n and start at $\overline{0}$.

Proof. Let $d = 2$ and $w = \langle v_0, v_1, \ldots, v_n \rangle$ be a walk in G_2 with length $n \in \mathbb{N}_0$. Let b_n be the number of walks w with $v_n = v_0 = \overline{0}$ and c_n be the number of walks w with $v_n = v_0 = \overline{0}$ but $v_j \neq \overline{0}$ for j with $0 < j < n$. By Proposition 5, these numbers do not depend on the start of the walk. We have $a_0 = c_0 = 0$ and $b_0 = d_0 = 1$. Clearly, $a_n \leq d_n$, $c_n \leq b_n \leq d_n$ and $d_n = 4^n$ for every $n \in \mathbb{N}_0$. We divide the walks counted by a_n into groups according to their first return to $\overline{0}$ at the vertex v_j. Using the relations $d_n = 4^n$ and $a_n \le 4^n$, we get

$$a_n = \sum_{j=0}^n c_j d_{n-j} \ \text{ hence } \ a_n/4^n = \sum_{j=0}^n c_j/4^j \le 1$$

for every $n \in \mathbb{N}_0$. It suffices to prove that

$$\sum_{j=0}^\infty c_j/4^j = 1\,.$$

We have for the generating functions $B(x) = \sum_{n\ge 0}(b_n/4^n)x^n = 1 + \dots$ and $C(x) = \sum_{n\ge 0}(c_n/4^n)x^n = \frac{x^2}{4} + \dots$ the equation

$$B(x) = \tfrac{1}{1-C(x)} = \sum_{k\ge 0} C(x)^k\,.$$

It holds formally, as a relation between formal power series: we divide a walk counted by b_n by the $k+1$ returns to $\overline{0}$ in k parts with lengths j_1, j_2, ..., j_k satisfying $j_1 + \dots + j_k = n$. These are counted by c_{j_1}, ..., c_{j_k}. It also holds at the level of real functions $B(x)$ and $C(x)$ for $x \in [0, 1)$, because both power series have radii of convergence ≥ 1 (since $b_n, c_n \le 4^n$).

Now it suffices to prove that

$$\lim_{x\to 1^-} B(x) = +\infty\ .$$

Indeed, then the above equation implies that $\lim_{x\to 1^-} C(x) = 1$ and this by Theorem 10 gives that

$$\sum_{j=0}^\infty (c_j/4^j) = C(1) = \lim_{x\to 1^-} C(x) = 1\,.$$

This is exactly the required sum of the infinite series.

In order to prove that $\lim_{x\to 1^-} B(x) = +\infty$, it suffices to prove by the Theorem 10 that

$$B(1) := \sum_{j=0}^\infty (b_j/4^j) = +\infty\,.$$

We prove it by finding a formula for b_n. Obviously $b_n = 0$ for odd n. For even lengths,

$$b_{2n} = \sum_{j=0}^{n} \frac{(2n)!}{j!\cdot(n-j)!\cdot j!\cdot(n-j)!} = \binom{2n}{n} \sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}^2 .$$

The first equality follows by considering all j steps to the right in the walk w. These force the same number of j steps to the left and the same number of $n-j$ steps up and down. These possibilities are counted by the multinomial coefficient $\binom{2n}{j,j,n-j,n-j}$. The last equality follows from the known binomial identity in Exercise 12. Stirling's formula for factorial approximation (Exercise 13) leads to the asymptotics $\binom{2n}{n} \sim cn^{-1/2}4^n$, for $n \to \infty$ a constant $c > 0$. So the $2n$-th summand in the series $B(1)$ is $\sim c^2 n^{-1}$ and

$$B(1) = \sum_{n=0}^{\infty}(b_n/4^n) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 4^{-2n} = +\infty$$

because $\sum n^{-1} = +\infty$. $\square$

Exercise 12 *Prove that for every $n \in \mathbb{N}_0$,*

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n} .$$

Hint: $\binom{n}{j} = \binom{n}{n-j}$ and $\binom{n}{j}$ is the number of j-element subsets of the n-element set.

Exercise 13 *Recall Stirling's formula*

$$n! \sim \sqrt{2\pi n}\big(n/\mathrm{e}\big)^n \quad (n \to \infty) .$$

Using integrals, prove the weak form $\sum_{m=1}^{n} \log m = n \log n - n + O(\log n)$ $(n \in \mathbb{N})$.

Proof. Let $d = 3$. Quantities a_n, b_n, c_n and d_n, generating functions $B(x)$ and $C(x)$, and sums of the series $B(1)$ and $C(1)$ are defined as in the previous proof, only now we are in the graph G_3 and the constant 4 is replaced by the constant 6. So now $B(x) = \sum_{n\geq 0}(b_n/6^n)x^n$ and $C(x) = \sum_{n\geq 0}(c_n/6^n)x^n$. Now we have to prove that

$$B(1) = \textstyle\sum_{n\geq 0}(b_n/6^n) < +\infty\,,$$

that is, that the series $B(1)$ converges. Then, since as before $B(x) = \frac{1}{1-C(x)}$ and by the Theorem 10 is $B(1) = \lim_{x\to 1^-} B(x)$ and $C(1) = \lim_{x\to 1^-} C(x)$, we get $C(1) = \lim_{x\to 1^-} C(x) < 1$. By this we are we are done because as before

$$C(1) = \textstyle\sum_{j=0}^{\infty}(c_j/6^j) = \lim_{n\to\infty}(a_n/6^n)\,.$$

We prove that $\sum_{n\geq 0} b_n/6^n$ converges. For odd n, $b_n = 0$ again. We estimate $b_{2n}/6^{2n}$ from above. For $n \in \mathbb{N}_0$, we have an upper bound

$$\begin{aligned} b_{2n}/6^{2n} &= \tfrac{1}{6^{2n}}\textstyle\sum_{\substack{j,\,k\in\mathbb{N}_0\\ j+k\leq n}} \tfrac{(2n)!}{j!\cdot j!\cdot k!\cdot k!\cdot(n-j-k)!\cdot(n-j-k)!}\\ &= \tbinom{2n}{n}4^{-n}\textstyle\sum_{\substack{j,\,k\in\mathbb{N}_0\\ j+k\leq n}}\left[\tfrac{1}{3^n}\tbinom{n}{j,\,k,\,n-j-k}\right]^2\\ &\leq \tbinom{2n}{n}4^{-n}\max_{\substack{x,\,y\ z\in\mathbb{N}_0\\ x+y+z=n}}\tfrac{1}{3^n}\tbinom{n}{x,\,y,\,z}\\ &= \tbinom{2n}{n}4^{-n}\tfrac{1}{3^n}\tbinom{n}{x_0,\,y_0,\,z_0}\,, \end{aligned}$$

where (x_0, y_0, z_0) is (m, m, m) when $n = 3m$, $(m+1, m, m)$ when $n = 3m+1$, and $(m+1, m+1, m)$ when $n = 3m+2$ $(m \in \mathbb{N}_0)$ – Exercise 14. On the first line, we counted as in the previous proof: j is the number of steps taken to the right, k is the number of steps up, and $n-j-k$ is the number of steps back. The second line represents a simple algebraic rearrangement. On the third line, we took advantage of the fact that by the multinomial expansion of

$3^n = (1+1+1)^n$, the numbers $[\dots]$ sum up to 1, and we used Exercise 15. On the fourth line, we found the maximum value of the trinomial coefficient with the help of Exercise 14.

By Stirling's formula, we have estimates

$$\tbinom{2n}{n} \ll \tfrac{4^n}{n^{1/2}} \ \text{ and } \ \tbinom{n}{x_0,\, y_0,\, z_0} \ll \tfrac{3^n}{n}\,.$$

Hence

$$b_{2n}/6^{2n} \ll n^{-1/2}\cdot n^{-1} = n^{-3/2}$$

and for some constant $c > 0$,

$$B(1) = \textstyle\sum_{n\ge 0}(b_n/6^n) = \sum_{n\ge 0}(b_{2n}/6^{2n}) < c\sum_{n=1}^{\infty}\frac{1}{n^{3/2}} < +\infty\,,$$

which we needed to show. □

Exercise 14 *Prove that for $a, b \in \mathbb{N}_0$ with $a \ge b+2$,*

$$\tfrac{1}{a!\cdot b!} \ge \tfrac{1}{(a-1)!\cdot(b+1)!}\,.$$

Exercise 15 *Let $A, a_1, \dots, a_n \ge 0$ be real numbers such that $a_i \le A$ and $a_1 + \dots + a_n = 1$. Then*

$$\textstyle\sum_{i=1}^{n} a_i^2 \le A\,.$$

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 1, 7, 11, 12 and 15.