MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2024/25
lecturer: Martin Klazar

LECTURE 7 (April 2, 2025) SOLVING THE BASEL PROBLEM BY FOURIER SERIES

• *The Basel problem.* What is the sum B of the series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\dots ?$$

According to the (English mutation of) Wikipedia, this problem was posed by Pietro Mengoli in 1650 and solved by Leonard Euler in 1734:

$$B=\frac{\pi^2}{6}\,.$$

The problem is named after Euler's hometown. There resided the clan of mathematicians Bernoulli's who also tried to solve the problem but they did not succeed.

• *Series.* We review basic notions of the theory of (infinite) series to make sense of the previous problem. A *series* $\sum a_n=\sum_{n=1}^{\infty}a_n$ is actually a sequence $(a_n)\subset\mathbb{R}$, to which we assign the sequence of *partial sums*

$$(s_n)=(a_1+a_2+\dots+a_n)\subset\mathbb{R}\,.$$

The *sum* of the series is $\lim s_n$, if this limit exists. If this limit is finite ($\in\mathbb{R}$), the series *converges*, else (the sum is $\pm\infty$ or does not exist) it *diverges*. Sums of series are denoted by the same symbols as the series themselves,

$$\sum a_n=\sum_{n=1}^{\infty}a_n=\lim s_n=\lim(a_1+a_2+\dots+a_n)\,.$$

In the following exercises we review basic results about series.

Exercise 1 (necessary condition for convergence) *If the series $\sum a_n$ converges then $\lim a_n = 0$.*

Exercise 2 *If the series $\sum a_n$ has almost all summands nonnegative, i.e. $n \ge n_0 \Rightarrow a_n \ge 0$, then $\sum a_n$ converges or has the sum $+\infty$.*

Exercise 3 (harmonic series) *The sum $\sum \frac{1}{n} = +\infty$.*

Exercise 4 *The sum $\sum \frac{1}{(n+1)n} = 1$.*

Exercise 5 *Using the previous exercise, prove that the series $\sum n^{-2}$ in the Basel problem converges.*

Exercise 6 (geometric series) *For $q \in (-1, 1)$, the sum*

$$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}\,.$$

Exercise 7 (Leibniz's criterion) *If $a_1 \ge a_2 \ge \cdots \ge 0$ and $\lim a_n = 0$, then the series $\sum (-1)^{n-1} a_n = a_1 - a_2 + a_3 - \ldots$ converges.*

Exercise 8 *Prove simply that if the sum*

$$\sum \frac{1}{n^2} = \frac{\pi^2}{6} \;\; \textit{then the sum} \;\; \sum \frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}\,.$$

• *Riemannian series.* We call a series $\sum a_n$ *Riemannian* if (i) $\lim a_n = 0$, (ii) $\sum a_{k_n} = +\infty$ and (iii) $\sum a_{z_n} = -\infty$, where (a_{k_n}), resp. (a_{z_n}), is the subsequence of nonnegative, resp. negative, summands in the series $\sum a_n$.

Exercise 9 (harder) *Fill in details in the sketch of the next proof.*

Theorem 10 (Riemann) *Let $\sum a_n$ be a Riemannian series. Then for every $S \in \mathbb{R}^*$ there is a bijection $\pi\colon \mathbb{N} \to \mathbb{N}$ such that*

$$\textstyle\sum_{n=1}^{\infty} a_{\pi(n)} = S\,.$$

Thus by reordering any Riemannian series we can get any sum.

Proof. Suppose that $\sum a_n$ is a Riemannian series and that $\sum a_{k_n}$ and $\sum a_{z_n}$ are as in the definition. We define π for any given $S \in \mathbb{R}$ (i.e., S is a real number, not $\pm\infty$) as follows. We initialize three variables by $i = 1$, $j = 0$ and $\pi(1) = k_1$. Suppose that $\pi(1)$, $\pi(2)$, $\dots$, $\pi(n)$ have been already defined and $a = \sum_{k=1}^{n} a_{\pi(k)}$. If $a < S$ then $i := i+1$, $j := j$ and $\pi(n+1) = k_i$. If $a \ge S$ then $i := i$, $j := j+1$ and $\pi(n+1) = z_j$. In this way we define a map $\pi\colon \mathbb{N} \to \mathbb{N}$. It follows that π is a bijection and

$$\textstyle\sum_{n=1}^{\infty} a_{\pi(n)} = S\,.$$

$\square$

- *Trigonometric series.* These are the series

$$\tfrac{a_0}{2} + \textstyle\sum_{n=1}^{\infty} \big(a_n \cos(nx) + b_n \sin(nx)\big)\,,$$

where $a_n, b_n \in \mathbb{R}$ are the *coefficients* and $x \in \mathbb{R}$ is a variable. One can view them as parametric systems of series parameterized by the variable x. Our goal is to derive expressions for a wide class of functions $f\colon [-\pi,\pi] \to \mathbb{R}$ as trigonometric series. Then we use it to solve the Basel problem.

Let $\mathcal{R}(-\pi,\pi)$ be the set of all Riemann integrable functions $f\colon [-\pi,\pi] \to \mathbb{R}$. For $f, g \in \mathcal{R}(-\pi,\pi)$ we define

$$\langle f,\, g\rangle = \textstyle\int_{-\pi}^{\pi} fg \quad (\in \mathbb{R})$$

(it follows from the theory of the Riemann integral that if f and g are in $\mathcal{R}(-\pi,\pi)$, then $fg \in \mathcal{R}(-\pi,\pi)$ too). It looks like a scalar product.

Exercise 11 *Prove that*

$$\langle f,\, g\rangle = \langle g,\, f\rangle \;,\;\; \langle f,\, f\rangle \geq 0$$

and, for $a, b \in \mathbb{R}$*,*

$$\langle af + bg,\, h\rangle = a\,\langle f,\, h\rangle + b\,\langle g,\, h\rangle \;\; .$$

But it is not completely a scalar product.

Exercise 12 *The equivalence*

$$\langle f,\, f\rangle = 0 \iff f \equiv 0$$

does not hold.

A function $f\colon \mathbb{R} \to \mathbb{R}$ is 2π*-periodic* if for every $x \in \mathbb{R}$ we have $f(x + 2\pi) = f(x)$.

Proposition 13 (orthogonality of sines and cosines) *For every two integers* $m, n \geq 0$*,*

$$\langle \sin(mx),\, \cos(nx)\rangle = 0 \; .$$

For every two integers $m, n \geq 0$*, except* $m = n = 0$*, one has that*

$$\langle \sin(mx),\, \sin(nx)\rangle = \langle \cos(mx),\, \cos(nx)\rangle = \begin{cases} \pi & \ldots \quad m = n \quad \textit{and} \\ 0 & \ldots \quad m \neq n \; . \end{cases}$$

Finally,

$$\langle \sin(0x),\, \sin(0x)\rangle = 0 \quad \textit{and} \quad \langle \cos(0x),\, \cos(0x)\rangle = 2\pi \; .$$

Proof. Let $m, n \in \mathbb{N}_0$. We compute the values

$$S_{m,n} = \langle \sin(mx), \sin(nx) \rangle, \; T_{m,n} = \langle \cos(mx), \cos(nx) \rangle$$

and

$$U_{m,n} = \langle \sin(mx), \cos(nx) \rangle \; .$$

Clearly, $S_{0,0} = 0$, $T_{0,0} = 2\pi$ and $U_{0,0} = 0$. Let m or n be non-zero, say $m \neq 0$ (for $n \neq 0$ the calculation is similar). Integration by parts using that $\sin(mx) = (-\cos(mx)/m)'$ and $\cos(mx) = (\sin(mx)/m)'$ yields

$$S_{m,n} = \frac{n}{m} \cdot T_{m,n}, \; T_{m,n} = \frac{n}{m} \cdot S_{m,n} \; \text{ and } \; U_{m,n} = -\frac{n}{m} \cdot U_{n,m}$$

– the first term $[\dots]_{-\pi}^{\pi}$ in the formula is always zero because ... is a 2π-periodic function. The first two equations together give

$$(1 - (n/m)^2) S_{m,n} = 0 = (1 - (n/m)^2) T_{m,n} \; .$$

If $n \neq m$ then $S_{m,n} = T_{m,n} = 0$. When $n = m$, then we know that $S_{m,m} = T_{m,m}$. But from the identity $\sin^2 x + \cos^2 x = 1$ (holding for every $x \in \mathbb{R}$) it follows that $S_{m,m} + T_{m,m} = \int_{-\pi}^{\pi} 1 = 2\pi$. Thus, $S_{m,m} = T_{m,m} = \pi$. The third equation above for $m = n$ gives $U_{m,m} = -U_{m,m}$ and so $U_{m,m} = 0$. To calculate $U_{m,n}$ for $m \neq n$, we express $U_{n,m}$ by integration by parts again using $\cos(mx) = (\sin(mx)/m)'$:

$$U_{n,m} = -(n/m) U_{m,n} \; .$$

Together $U_{m,n} = (n/m)^2 U_{m,n}$ and again $U_{m,n} = 0$. In summary: $S_{m,m} = T_{m,m} = \pi$ for $m \in \mathbb{N}$, $S_{0,0} = 0$ and $T_{0,0} = 2\pi$, and all other values of $S_{m,n}$, $T_{m,n}$ and $U_{m,n}$ for $m, n \in \mathbb{N}_0$ are zero. □

• *The Fourier series of a function.* For $f \in \mathcal{R}(-\pi, \pi)$ we define the *cosine Fourier coefficients* by

$$a_n = \frac{\langle f(x), \cos(nx) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, \mathrm{dx}, \; n = 0, 1, \dots$$

and its *sine Fourier coefficients* by

$$b_n = \frac{\langle f(x), \sin(nx)\rangle}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\sin(nx)\,\mathrm{dx},\ \ n = 1,\,2,\,\ldots\,.$$

The *Fourier series of the function* f $(\in \mathcal{R}(-\pi,\pi))$ is the trigonometric series

$$F_f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\big(a_n\cos(nx) + b_n\sin(nx)\big)\,,$$

where a_n and b_n are, respectively, the cosine and sine Fourier coefficients of f. From the perspective of functional analysis, we work in an infinite-dimensional vector space with the (almost) scalar product $\langle\cdot,\cdot\rangle$, in which the "coordinate axes" (elements of the orthogonal basis) are the functions

$$\{\cos(nx):\ \ n\in\mathbb{N}_0\}\cup\{\sin(nx):\ \ n\in\mathbb{N}\}\,.$$

Fourier coefficients of a given function f are the coordinates of f with respect to these infinitely many coordinate axes. In contrast with Cartesian coordinates of points in $\mathbb{R}^n$, not every function is equal to the sum of its Fourier series. In a moment we present sufficient conditions for this to hold.

- *Bessel's inequality.*

Theorem 14 (Bessel's Inequality) *For every function f in $\mathcal{R}(-\pi,\pi)$ the Fourier coefficients a_n and b_n satisfy the inequality*

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \le \frac{\langle f,f\rangle}{\pi} = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,.$$

Proof. We denote by $s_n = s_n(x)$, $n = 1, 2, \ldots$, the n-th partial sum of the Fourier series of the function f:

$$\begin{aligned} s_n &= \frac{a_0}{2} + \sum_{k=1}^{n}\big(a_k\cos(kx) + b_k\sin(kx)\big) \\ &= \sum_{k=0}^{n}\big(a_k'\cos(kx) + b_k'\sin(kx)\big)\,, \end{aligned}$$

where

$$a_k = \pi^{-1}\langle f,\, \cos(kx)\rangle,\; b_k = \pi^{-1}\langle f,\, \sin(kx)\rangle,\; k = 0,\, 1,\, 2,\, \dots\,,$$

$a'_0 = \frac{a_0}{2}$, $a'_k = a_k$ for $k > 0$, $b'_0 = 0$ and $b'_k = b_k$ for $k > 0$. Due to the linearity of the (almost) scalar product $\langle\cdot,\cdot\rangle$, the definition of the numbers a'_k, b'_k, a_k, b_k and the orthogonality of functions $\sin(kx)$ and $\cos(kx)$, the (almost) scalar product $\langle s_n, s_n\rangle$ equals to

$$\begin{aligned}&\textstyle\sum_{k=0}^{n}\big((a'_k)^2\langle\cos(kx),\, \cos(kx)\rangle + (b'_k)^2\langle\sin(kx),\, \sin(kx)\rangle\big)\\ &= \pi\big(\frac{a_0^2}{2} + \textstyle\sum_{k=1}^{n}(a_k^2 + b_k^2)\big)\,.\end{aligned}$$

Also,

$$\begin{aligned}\langle s_n,\, f\rangle &= \textstyle\sum_{k=0}^{n}\big(a'_k\langle\cos(kx),\, f\rangle + b'_k\langle\sin(kx),\, f\rangle\big)\\ &= \pi\big(\frac{a_0^2}{2} + \textstyle\sum_{k=1}^{n}(a_k^2 + b_k^2)\big)\,.\end{aligned}$$

On the other hand,

$$0 \le \langle f - s_n,\, f - s_n\rangle = \langle f,\, f\rangle - 2\langle s_n,\, f\rangle + \langle s_n,\, s_n\rangle\,,$$

hence $2\langle s_n, f\rangle - \langle s_n, s_n\rangle \le \langle f, f\rangle$. Thus for every n,

$$\frac{a_0^2}{2} + \textstyle\sum_{k=1}^{n}(a_k^2 + b_k^2) = \frac{2\langle s_n,f\rangle - \langle s_n,s_n\rangle}{\pi} \le \frac{\langle f,f\rangle}{\pi}\,.$$

The series of squares of the Fourier coefficients of the function f converges and its sum is bounded by the stated value. □

Exercise 15 (Riemann–Lebesgue Lemma) *Using Bessel's inequality, prove that for every function $f \in \mathcal{R}(-\pi,\pi)$*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{dx} = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{dx} = 0\,.$$

(Hint: see Exercise 1).

• *Piece-wise smooth functions and Dirichlet's theorem.* The function

$$f\colon [a,\, b] \to \mathbb{R} \ ,$$

where $a < b$ are real numbers, is *piece-wise smooth* if there is a partition

$$a = a_0 < a_1 < a_2 < \cdots < a_k = b,\ \ k \in \mathbb{N} \ ,$$

of the interval $[a, b]$ such that on every interval (a_{i-1}, a_i), $i = 1, 2, \ldots, k$, f has continuous derivative f', for every $i = 1, 2, \ldots, k$ there exist finite one-sided limits

$$f(a_i - 0) = \lim_{x \to a_i^-} f(x) \ \text{ and } \ f'(a_i - 0) = \lim_{x \to a_i^-} f'(x)$$

and for each $i = 0, 1, \ldots, k-1$ there exist finite one-sided limits

$$f(a_i + 0) = \lim_{x \to a_i^+} f(x) \ \text{ and } \ f'(a_i + 0) = \lim_{x \to a_i^+} f'(x) \ .$$

A piece-wise smooth function can be at several points in the interval $[a, b]$ discontinuous, but at the points of discontinuity it has finite one-sided limits and one-sided non-vertical tangents.

Exercise 16 *Is the function* $f\colon [-1, 1] \to \mathbb{R}$, *defined as* $f(x) = (-x)^{1/3}$ *for* $x \in [-1, 0]$ *and* $f(x) = x^{1/3}$ *for* $x \in [0, 1]$, *piece-wise smooth?*

Exercise 17 *Is the signum function* $\mathrm{sgn}\colon [-1, 1] \to \mathbb{R}$, *defined as* $\mathrm{sgn}(x) = -1$ *for* $x \in [-1, 0)$, $\mathrm{sgn}(0) = 0$ *and* $\mathrm{sgn}(x) = 1$ *for* $x \in (0, 1]$, *piece-wise smooth?*

Theorem 18 (Dirichlet's) *Let*

$$f\colon \mathbb{R} \to \mathbb{R}$$

be a 2π-periodic function such that its restriction to the interval $[-\pi, \pi]$ is piece-wise smooth. Then for every $a \in \mathbb{R}$ its Fourier series $F_f(x)$ sums to

$$F_f(a) = \tfrac{f(a+0)+f(a-0)}{2} := \tfrac{\lim_{x\to a^+} f(x)+\lim_{x\to a^-} f(x)}{2} \ .$$

Thus, at each point of continuity $a \in \mathbb{R}$ of the function $f(x)$, its Fourier series sums to the functional value, $F_f(a) = f(a)$.

Proof. We will probably skip it. □

We say that the function $f\colon [a, b] \to \mathbb{R}$ is *smooth* if it has on (a, b) continuous derivative f' and at the ends a and b the functions $f(x)$ and $f'(x)$ have finite limits.

Corollary 19 (on smooth function) *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a 2π-periodic and continuous function whose restriction to the interval $[-\pi, \pi]$ is smooth. Then for each $a \in \mathbb{R}$ is*

$$F_f(a) = f(a) \ .$$

Any continuous and smooth function is therefore equal to the sum of its Fourier series.

Proof. This follows from the previous theorem: by the assumption f is continuous on $\mathbb{R}$. □

• *Back to the Basel problem.* Let $I \subset \mathbb{R}$ be an interval symmetric with respect to the origin and $f\colon I \to \mathbb{R}$. We say that the function f is *even* (resp. *odd*) if for every $x \in I$, $f(-x) = f(x)$ (resp. $f(-x) = -f(x)$).

Exercise 20 *Let $f \in \mathcal{R}(-\pi, \pi)$. Prove that all sine (or cosine) Fourier coefficients of an even (or odd) functions f are zero.*

How do you simplify cosine (or sine) Fourier coefficients of an even (or odd) function?

We calculate the Fourier series of the function $f\colon \mathbb{R} \to \mathbb{R}$ defined on the interval $[-\pi, \pi]$ by $f(x) = x^2$ and 2π-periodically extended to the entire $\mathbb{R}$ (which is possible due to the fact that $(-\pi)^2 = \pi^2$). Its sine Fourier coefficients are zero according to the previous exercise. The first (actually zeroth) cosine Fourier coefficient is (according to this exercise)

$$a_0 = \tfrac{2}{\pi} \int_0^\pi x^2 \,\mathrm{dx} = \tfrac{2\pi^2}{3}\,.$$

Next ($n \in \mathbb{N}$)

$$\begin{aligned} a_n &= \tfrac{2}{\pi} \int_0^\pi x^2 \overbrace{\cos(nx)}^{(\sin(nx)/n)'} \,\mathrm{dx} \\ &= \tfrac{2}{\pi n} \underbrace{\left[x^2 \sin(nx)\right]_0^\pi}_{0-0=0} - \tfrac{4}{\pi n} \int_0^\pi x \overbrace{\sin(nx)}^{(-\cos(nx)/n)'} \,\mathrm{dx} \\ &= \tfrac{4}{\pi n^2} \underbrace{\left[x \cos(nx)\right]_0^\pi}_{\pi(-1)^n} - \tfrac{4}{\pi n^2} \underbrace{\int_0^\pi \cos(nx)\,\mathrm{dx}}_{0-0=0} = (-1)^n \tfrac{4}{n^2}\,. \end{aligned}$$

Since the function f is continuous and smooth on $[-\pi, \pi]$, by Corollary 19 one has for every $a \in \mathbb{R}$ that

$$f(a) = \tfrac{a_0}{2} + \textstyle\sum_{n=1}^\infty a_n \cos(na) = \tfrac{\pi^2}{3} + 4 \sum_{n=1}^\infty (-1)^n \tfrac{\cos(na)}{n^2}\,.$$

For $a = \pi$ we get

$$\pi^2 = f(\pi) = \tfrac{\pi^2}{3} + 4 \textstyle\sum_{n=1}^\infty (-1)^n \tfrac{(-1)^n}{n^2}, \quad \text{so that} \quad \sum_{n=1}^\infty \tfrac{1}{n^2} = \tfrac{\pi^2}{6}\,.$$

Exercise 21 *The function $f(x)$ is defined on the interval $[-\pi, \pi)$ as $f(x) = \pi - x$ and is 2π-periodically extended to $\mathbb{R}$. Expand it into Fourier series.*

Exercise 22 *What sum of the infinite series do we get from the previous expansion (using Dirichlet's theorem) for* $x = \frac{\pi}{2}$*?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 1, 8, 9, 16 and 20.