MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2025/26
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LECTURE 6 (March 23, 2026)
APPLICATIONS OF BAIRE'S THEOREM:
NON-DIFFERENTIABLE FUNCTIONS AND
GROWTHS OF PERMUTATIONS

• *Non-differentiable continuous functions.* Let $I = [0,1]$. By $C(I)$ we denote the set of continuous functions $f\colon I \to \mathbb{R}$. Recall that for $x \in \mathbb{R}$ and $\delta > 0$,

$$P(x,\,\delta) = (x-\delta,\,x+\delta) \setminus \{x\} = (x-\delta,\,x) \cup (x,\,x+\delta)$$

is the deleted δ-neighborhood of x. We prove with the help of the Baire theorem the following.

Theorem 1 *There exists a function $f \in C(I)$ such that for every $x \in I$ and every $\delta > 0$,*

$$\sup\left(\left\{\left|\tfrac{f(y)-f(x)}{y-x}\right|\colon\ y \in P(x,\,\delta) \cap I\right\}\right) = +\infty\,.$$

Recall that $f\colon I \to \mathbb{R}$ is differentiable at $x \in I$ if $f'(x) \in \mathbb{R}$.

Exercise 2 *The function f in* Theorem 1 *is continuous on I but it is not differentiable at any point of I.*

• *Four lemmas.* We prove Theorem 1 with the help of four lemmas.

Lemma 3 (lemma a) *If $f \in C(I)$ is such that for every point $x \in I$,*

$$\sup\left(\left\{\left|\tfrac{f(y)-f(x)}{y-x}\right|\colon\ y \in I \setminus \{x\}\right\}\right) = +\infty\,,$$

then f has the property in Theorem 1*; the parameter δ in* Theorem 1 *is superfluous.*

Proof. We assume that $f \in C(I)$ is as stated. For every $x \in I$ and $\delta > 0$, the set

$$Q(x,\, \delta) = I \setminus U(x,\, \delta) = [0,\, 1] \setminus (x - \delta,\, x + \delta)$$

is compact (Exercise 4). Let $M(x, \delta)$ be the maximum value of the continuous function

$$Q(x,\, \delta) \ni y \mapsto \big|(f(y) - f(x))/(y - x)\big| \geq 0\,.$$

Let $x \in I$ and $\delta > 0$ be given. By the assumption there exists $y \in I \setminus \{x\}$ such that

$$\left|\tfrac{f(y)-f(x)}{y-x}\right| > M(x, \delta)\,.$$

Then $y \notin Q(x, \delta)$ and $y \in P(x, \delta)$. We see that f has the property in Theorem 1. $\square$

Exercise 4 *The set $Q(x, \delta)$ is compact.*

Exercise 5 *Why is the function $y \mapsto \left|\frac{f(y)-f(x)}{y-x}\right|$ continuous?*

Recall for any set X the infinity norm

$$\|f\|_\infty = \sup(\{|f(x)| \colon\ x \in X\})$$

on the set B of bounded functions $f \colon X \to \mathbb{R}$. It makes B a metric space $(B, \|f - g\|_\infty)$.

Exercise 6 *Show that this is a metric space.*

Lemma 7 (lemma b) *Let (M, d) be a metric space, let $(x_n) \subset M$ have $\lim x_n = x_0$, and let $f_n \colon M \to \mathbb{R}$ be a sequence of functions converging in $\|\cdot\|_\infty$ to a function $f \colon M \to \mathbb{R}$ that is continuous. Then*

$$\lim_{n\to\infty} f_n(x_n) = f(x_0)\,.$$

Proof. By TI,

$$|f_n(x_n) - f(x_0)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x_0)| \,.$$

Let $\varepsilon > 0$ be given. We can make the first $|\cdot|$ on the RHS $< \frac{\varepsilon}{2}$ for every $n \geq n_0$ because $\|f_n - f\|_\infty \to 0$. The same holds for the second $|\cdot|$ on the right side for every $n \geq n_1$, due to Heine's definition of continuity of f at the point x_0. Hence $n \geq \max(n_0, n_1)$ $\Rightarrow |f_n(x_n) - f(x_0)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. □

A *broken line* going through the points (a_0, b_0), (a_1, b_1), ..., (a_k, b_k) in $\mathbb{R}^2$ in this order, where $a_0 < a_1 < \cdots < a_k$, is the function $f\colon [a_0, a_k] \to \mathbb{R}$ which is on every interval $[a_{i-1}, a_i]$, $i = 1, 2, \ldots, k$, defined by

$$f(x) = \tfrac{(b_i - b_{i-1})(x - a_{i-1})}{a_i - a_{i-1}} + b_{i-1} \,.$$

Thus $f(a_{i-1}) = b_{i-1}$ and $f(a_i) = b_i$. Its graph on the interval $[a_{i-1}, a_i]$ is the straight segment joining the points (a_{i-1}, b_{i-1}) and (a_i, b_i). We call these segments just *segments*.

Exercise 8 *Every broken line is a continuous function.*

Recall that the *slope* of a line $y = ax + b$ is the number a. The *slope* of a segment is the slope of the line extending the segment. The *secant* of a function $f\colon M \to \mathbb{R}$, where $M \subset \mathbb{R}$, is any line going through two distinct points on the graph of f.

Lemma 9 (lemma c) *Let $\varepsilon > 0$ and $f \in C(I)$. Then there is a function $g \in C(I)$ and a constant $M > 0$ such that the following holds.*

1. $\|f - g\|_\infty < \varepsilon$.

2. If $x, y \in I$ and $x \neq y$, then $\left|\frac{g(y)-g(x)}{y-x}\right| \leq M$.

Proof. Let $f \in C(I)$ and let an $\varepsilon > 0$ be given. Since I is compact, f is uniformly continuous (Exercise 10). So for every sufficiently large m and every $i = 0, 1, \ldots, m$ it holds that

$$\tfrac{i}{m} \leq x \leq \tfrac{i+1}{m} \Rightarrow |f(\tfrac{i}{m}) - f(x)|,\ |f(\tfrac{i+1}{m}) - f(x)| < \tfrac{\varepsilon}{2}\,.$$

We draw a broken line g through the points $(i/m, f(i/m))$, $i = 0, 1, \ldots, m$. For g the above implication holds too and with the same m (Exercise 11). Thus for every $x \in I$,

$$|f(x) - g(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

(Exercise 12). Thus, g has property 1. For every $x, y \in I$ with $x \neq y$,

$$\left|\tfrac{g(y)-g(x)}{y-x}\right| \leq |s|$$

where s is the largest, in absolute value, slope of a segment of the broken line g (Exercise 13). We see that g has property 2. □

Exercise 10 *Why is any $f \in C(I)$ uniformly continuous?*

Exercise 11 *Show that the displayed implication holds for the broken line g.*

Exercise 12 *Prove the displayed inequality.*

Exercise 13 *Prove the displayed inequality.*

Lemma 14 (lemma d) *Let $\varepsilon > 0$ and $T > 0$. Then there is $g \in C(I)$ such that the following holds.*

1. $\|g\|_\infty < \varepsilon$.

2. For every $x \in I$ we have $\left|\frac{g(y)-g(x)}{y-x}\right| > T$ for some $y \in I\backslash\{x\}$.

Proof. Let $\varepsilon > 0$ and $T > 0$ be given. We take a large even number $m \in \mathbb{N}$ such that $2m\varepsilon/3 > T$, take $m+1$ plane points

$$\big(i/m,\ (\varepsilon/3)(1-(-1)^i\big) \in \mathbb{R}^2,\ i = 0,\ 1\ \ldots,\ m\,,$$

and draw through them a broken line g. It joins $(0,0)$ and $(1,0)$ and has $m/2$ hills, each with height $2\varepsilon/3$ and base width $2/m$. Thus $\|g\|_\infty = 2\varepsilon/3 < \varepsilon$ and condition 1 holds. Let u be a point on the graph of g. We lead through it the secant line extending the segment containing u (if u lies in two segments, we choose any of them). It has a slope $> T$ because both sides of any hill have $|\text{slope}| = \frac{2\varepsilon/3}{1/m} = \frac{2m\varepsilon}{3} > T$. Condition 2 holds too. □

• *Proof of* Theorem 1. We show that there is a continuous function $f\colon I \to \mathbb{R}$ that is not differentiable at any point of I.

Proof of Theorem 1. For $n \in \mathbb{N}$ we define sets

$$A_n = \left\{f \in C(I)\colon\ \exists\, x \in I\ \forall\, y \in I \setminus \{x\}\colon \left|\tfrac{f(y)-f(x)}{y-x}\right| \le n\right\}.$$

We show that every set A_n is a sparse subset of the metric space $(C(I), \|f-g\|_\infty)$. By this we will be done. By Proposition 17 below this metric space is complete. By Baire's theorem there exists a function

$$f \in C(I) \setminus \textstyle\bigcup_{n=1}^{\infty} A_n\,.$$

Thus the function f is continuous and has the property described in Lemma 3. By this lemma, f has the property in Theorem 1 and by Exercise 2 the function f is not differentiable at any point of I.

We show that every set $A_n \subset C(I)$ is closed and contains no ball: for every ball $B(f,r)$ in the metric space, $B(f,r) \not\subset A_n$. It follows from this that A_n is a sparse set (Exercise 15).

We show that A_n is closed to limits of sequences. Let $(f_k) \subset A_n$ with $\lim_{k\to\infty} f_k = f \in C(I)$. We show that $f \in A_n$. Since $f_k \in A_n$, there is a number $x_k \in I$ such that for every $y \in I \setminus \{x_k\}$,

$$\left|\tfrac{f_k(y)-f_k(x_k)}{y-x_k}\right| \le n\,.$$

As we know, (x_k) has a convergent subsequence with a limit in I. To simplify notation, we assume that already $\lim_{k\to\infty} x_k = x_0 \in I$. For every $y \in I \setminus \{x_0\}$ we have, by the property of the point x_k and Lemma 7, that

$$n \ge \lim_{k\to\infty} \left|\tfrac{f_k(y)-f_k(x_k)}{y-x_k}\right| = \left|\tfrac{f(y)-f(x_0)}{y-x_0}\right|$$

(non-strict inequalities are preserved in limits). The number x_0 therefore witnesses that $f \in A_n$ and A_n is a closed set.

Let $B(f, r) \subset C(I)$ be a ball. We find a point $g \in B(f, r) \setminus A_n$. We define it as $g = g_1 + g_2$; we get the functions g_1 and g_2 using Lemmas 9 and 14, respectively. By Lemma 9 we have a function $g_1 \in C(I)$ and a constant $M > 0$ such that $\|f - g_1\|_\infty < r/2$ and that all secants of the graph of g_1 have slope in absolute value $\le M$. Then, by Lemma 14, we have a function $g_2 \in C(I)$ such that $\|g_2\|_\infty < r/2$ and that through every point in the graph of g_2 there goes a secant line with slope in absolute value $> M + n$. By the triangle inequality,

$$\|f - g\|_\infty \le \|f - g_1\|_\infty + \|g_2\|_\infty < \tfrac{r}{2} + \tfrac{r}{2} = r$$

and $g \in B(f, r)$. Let $x \in I$ be arbitrary. By the property of the function g_2 we take a $y \in I \setminus \{x\}$ such that $\left|\tfrac{g_2(y)-g_2(x)}{y-x}\right| > M + n$.

Then

$$\begin{aligned}\left|\tfrac{g(y)-g(x)}{y-x}\right| &= \left|\tfrac{g_2(y)-g_2(x)}{y-x}+\tfrac{g_1(y)-g_1(x)}{y-x}\right|\\ &\geq \left|\tfrac{g_2(y)-g_2(x)}{y-x}\right|-\left|\tfrac{g_1(y)-g_1(x)}{y-x}\right|\\ &> (M+n)-M=n\end{aligned}$$

and $g \notin A_n$. On the first line we used the definition of g, on the second the inequality from Exercise 16 and on the third the properties of the functions g_1 and g_2. □

Exercise 15 *Prove that every closed set X, in a metric space, with empty interior (X contains no ball) is sparse.*

Exercise 16 *Prove that for every two real numbers a and b,*

$$|a-b| \geq |a|-|b| \ .$$

- *Completeness of the metric space of continuous functions.*

Proposition 17 *The metric space $(C(I), \|f-g\|_\infty)$ is complete.*

Proof. Let $(f_n) \subset C(I)$ be a Cauchy sequence, so that for every $\varepsilon > 0$ there is $m \in \mathbb{N}$ such that

$$n,\, n' \geq m \Rightarrow \|f_n - f_{n'}\|_\infty < \varepsilon \ .$$

Then for every $x \in I$ the sequence $(f_n(x)) \subset \mathbb{R}$ is Cauchy, therefore convergent, and we can define

$$f(x) = \lim_{n\to\infty} f_n(x) \ .$$

Thus we have a function $f\colon I \to \mathbb{R}$ such that pointwisely $f_n \to f$. We prove that the convergence is uniform: $\|f-f_n\|_\infty \to 0$. Let

$x \in I$ and $\varepsilon > 0$ be given. We take an m (independent of x) such that the above displayed Cauchy condition holds with $\varepsilon/2$. Then we take $k \geq m$ such that $|f_k(x) - f(x)| < \frac{\varepsilon}{2}$. Thus if $n \geq m$, then

$$|f_n(x) - f(x)| \leq |f_n(x) - f_k(x)| + |f_k(x) - f(x)| < \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon$$

and $\lim f_n = f$ in the metric space $C(I)$.

It remains to show that $f \in C(I)$, that is, f is continuous. Let $x_0 \in I$ and $\varepsilon > 0$ be given. We take n_0 such that

$$n \geq n_0 \Rightarrow \|f - f_n\|_\infty \leq \tfrac{\varepsilon}{2}\,.$$

We take a $\delta > 0$ such that

$$x \in U(x_0,\, \delta) \cap I \Rightarrow |f_{n_0}(x) - f_{n_0}(x_0)| \leq \tfrac{\varepsilon}{2}$$

(we use the continuity of f_{n_0} at x_0). Then for every $x \in U(x_0, \delta) \cap I$ we have

$$\begin{aligned} &f(x) - f(x_0)| \leq \\ &|f(x) - f_{n_0}(x)| + |f_{n_0}(x) - f_{n_0}(x_0)| \leq \varepsilon/2 + \varepsilon/2 = \varepsilon\,. \end{aligned}$$

□

Now the proof of Theorem 1 is complete.

• *Baire's theorem and enumeration of permutations.* For $m \leq n$ in $\mathbb{N}$ $(= \{1, 2, \dots\})$ and two permutations $\pi\colon [m] \to [m]$ and $\rho\colon [n] \to [n]$ we write $\pi \preceq \rho$, and say that π *is contained in* ρ, if there exist numbers $i_1 < i_2 < \cdots < i_m$ in $[n]$ such that for every $j, k \in [m]$,

$$\pi(j) < \pi(k) \iff \rho(i_j) < \rho(i_k)\,.$$

Let $\mathcal{S}$ be the set of all finite permutations $\pi\colon [n] \to [n]$, $n \in \mathbb{N}$, and let $\mathcal{S}_n \subset \mathcal{S}$ be the ($n!$-element) set of permutations of $[n]$.

Exercise 18 *Show that* $(\mathcal{S}, \preceq)$ *is a non-strict partial order.*

We say that a set $X \subset \mathcal{S}$ is a *permutation class* if for every two permutations π and ρ,

$$\pi \preceq \rho \in X \Rightarrow \pi \in X \ .$$

In the last twenty years, many results on enumeration of permutation classes X,which means on the counting functions of the form

$$n \mapsto |X \cap \mathcal{S}_n|$$

($|A|$ denotes the cardinality of a finite set A), were obtained. Basic result is the next theorem.

Theorem 19 (A. Marcus and G. Tardos, 2004) *Let* X *be a permutation class different from* $\mathcal{S}$*. Then there is a constant* $c \in \mathbb{N}$ *such that*

$$|X \cap \mathcal{S}_n| \le c^n$$

for every $n \in \mathbb{N}$*.*

Exercise 20 *Let* $\pi \in \mathcal{S}_2$ *be the identical permutation, so that* $\pi(1) = 1$ *and* $\pi(2) = 2$*, and let* X *be any permutation class such that* $\pi \notin X$*. Then* $|X \cap \mathcal{S}_n| \le 1$ *for every* n*.*

By the Marcus–Tardos theorem, for every permutation class X different from $\mathcal{S}$ one can define its finite *growth rate*

$$c(X) = \limsup_{n \to \infty} |X \cap \mathcal{S}_n|^{1/n} \ \ (< +\infty) \, .$$

For example, it is known that $c(\{\rho \in \mathcal{S} \mid \rho \not\succeq \pi\}) = 4$ for every $\pi \in \mathcal{S}_3$. In fact,

$$|X \cap \mathcal{S}_n| = \tfrac{1}{n+1}\binom{2n}{n}$$

for every n for any of these six permutations classes X.

For some time there was a conjecture that every growth rate of a permutation class is an algebraic number. It was refuted by the following result.

Theorem 21 (M. Albert and S. Linton, 2009) *There is a nonempty closed set $A \subset [0, +\infty)$ such that A has no isolated point and every number in A is the growth rate of a permutation class.*

As we saw in the lecture before the last lecture, Baire's theorem implies that each such set A is uncountable. Thus we have uncountably many growth rates of permutation classes, and (since the set of algebraic numbers is countable) almost all of them are non-algebraic.

Corollary 22 (transcendental growths) *There exist growth rates of permutation classes that are non-algebraic.*

Exercise 23 *How does it exactly follow from Baire's theorem that the above set A is uncountable?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 2, 4, 15, 20 and 23.