

MATHEMATICAL ANALYSIS 3 (NMAI056)

summer term 2024/25

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**LECTURE 5 (March 19, 2025) THE WEAK JORDAN
CIRCUIT THEOREM. THOMASSEN'S PROOF**

I show you a proof following the remarkable proof *Carsten Thomassen (1948)* gave in 1992 for the famous theorem of *Camille Jordan (1838–1922)* in topology of the plane. We prove that any topological circuit disconnects \mathbb{R}^2 . I rely on the article The Jordan Curve Theorem, Formally and Informally, *Amer. Math. Monthly* **114** (2007), 882–894 by *Thomas C. Hales (1958)*.

Exercise 1 *Learn about achievements of these three excellent mathematicians.*

• *Arcs, circuits, PL maps and two theorems.* Let $I = I_{a,b} = [a, b]$ where $a < b$ are in \mathbb{R} . An *arc* is an injective continuous map

$$f: I \rightarrow \mathbb{R}^2.$$

Here I is a subspace of the MS (\mathbb{R}, e_1) and \mathbb{R}^2 is the MS (\mathbb{R}^2, e_2) . The points $f(a)$ and $f(b)$ in \mathbb{R}^2 are the *endpoints of f* . We say that *f joins $f(a)$ to $f(b)$* . If $f[I] \subset X \subset \mathbb{R}^2$, we say that *f joins $f(a)$ to $f(b)$ in X* . The set $f^0 = f[I]^0 = f[(a, b)]$ is the *interior (of the arc f)*. The *distance between two sets $A, B \subset \mathbb{R}^2$* is the infimum

$$e_2(A, B) = \inf(\{e_2(\bar{x}, \bar{y}) \mid \bar{x} \in A, \bar{y} \in B\}).$$

If A and B are disjoint and compact, their distance is positive.

A *circuit* is any continuous map $f: I = I_{a,b} \rightarrow \mathbb{R}^2$ such that $f(x) \neq f(y)$ for $x \neq y$, with the exception of $f(a) = f(b)$. We extend f to the $(b - a)$ -periodic function $f_e: \mathbb{R} \rightarrow \mathbb{R}^2$.

A map $f: I \rightarrow \mathbb{R}^2$ is PL (*piece-wise linear*) if there exist a partition $a = t_0 < t_1 < \dots < t_n = b$, $n \in \mathbb{N}$, of I and $2n$ vectors $\overline{s}_i, \overline{c}_i \in \mathbb{R}^2$, $i = 1, \dots, n$, such that no $\overline{s}_i = (0, 0)$ and

$$\forall i \in [n] \forall t \in [t_{i-1}, t_i] (f(t) = t \cdot \overline{s}_i + \overline{c}_i) .$$

The points $f(t_i) \in \mathbb{R}^2$, $i = 0, 1, \dots, n$, are the *corners (of f)*. The plane straight segments $f[[t_{i-1}, t_i]]$ are the *segments (of f)*. For any two points $\overline{p}, \overline{q} \in \mathbb{R}^2$ we denote by $s(\overline{p}, \overline{q})$ the straight segment joining them. An *axis* of a corner $f(t_i)$ with $0 < i < n$ is the line going through $f(t_i)$ that halves both angles at $f(t_i)$ (determined by the two segments of f incident with the corner). If $f(t_0) = f(t_n)$, this corner also has an axis. An *oriented PL circuit* is one where all segments are oriented consistently in one of the two ways.

Exercise 2 *Show that any PL map is continuous.*

Two famous theorems about images of arcs and circuits are

Theorem 3 (the Arc Theorem) *For any arc $f: I \rightarrow \mathbb{R}^2$ the set $\mathbb{R}^2 \setminus f[I]$ is connected.*

and (<https://kam.mff.cuni.cz/~klazar/JordanPic1.pdf>)

Theorem 4 (the Weak Jordan Theorem) *For any circuit $f: I \rightarrow \mathbb{R}^2$ the set $\mathbb{R}^2 \setminus f[I]$ is disconnected.*

We prove the latter theorem. We discuss the full Jordan Theorem in the concluding remarks. An outline of a proof of the Arc Theorem is given in [Hales] on pp. 890–891.

Exercise 5 *In the two theorems, the complements are open plane sets.*

• *Two results on connected sets.* Recall that $X \subset \mathbb{R}^2$ is connected if $\neg \exists$ open (or closed) sets $A, B \subset \mathbb{R}^2$ cutting X , i.e., such that $X \subset A \cup B$, both intersections $X \cap A$ and $X \cap B$ are nonempty and $X \cap A \cap B = \emptyset$. A set $X \subset \mathbb{R}^2$ is *PL-connected* if for any $x, y \in X$, $x \neq y$, there is a PL arc $f: I \rightarrow \mathbb{R}^2$ joining x to y in X .

Theorem 6 (conn. \leftrightarrow PL-conn.) *Any open set $X \subset \mathbb{R}^2$ is connected iff it is PL-connected.*

Proof. $\neg \Rightarrow \neg$. Suppose that A and B cut X , $x \in X \cap A$, $y \in X \cap B$ and that the PL arc $f: I \rightarrow \mathbb{R}^2$ with $f[I] \subset X$ joins x to y . Then A and B cut $f[I]$ and $f[I]$ is disconnected. This is impossible because $f[I]$ is connected as a continuous image of the connected interval I . Hence f does not exist.

$\neg \Leftarrow \neg$. Consider the partition X/\sim of X by the equivalence relation \sim on X (Exercise 8) defined by $x \sim y$ iff a PL arc joins x to y in X . It is not hard to see that every block $A \in X/\sim$ is an open set. We assume that X is not PL-connected: $|X/\sim| \geq 2$. We take any block $A \in X/\sim$ and define B to be the union of all other blocks. Then A and B are open sets cutting X . Thus X is disconnected. \square

Exercise 7 *Describe a set $X \subset \mathbb{R}^2$ that is a countable union of plane segments and is connected but is not PL-connected.*

Exercise 8 *Show that the relation \sim in the previous proof is transitive. Or, better, prove this more general proposition.*

Proposition 9 (PL maps and PL arcs) *For every PL map $f: I = I_{a,b} \rightarrow \mathbb{R}^2$ there is a PL arc $g: I' \rightarrow \mathbb{R}^2$ such that $g[I'] \subset f[I]$ and g joins $f(a)$ to $f(b)$.*

• (PL) *configurations*. A *C-configuration*, abbreviated *C-conf*, is any circuit $f: I \rightarrow \mathbb{R}^2$ such that $\mathbb{R}^2 \setminus f[I]$ is connected. Our goal is to prove that no *C-conf* exists, i.e., that Theorem 4 holds. A *$K_{3,3}$ -configuration*, abbreviated *$K_{3,3}$ -conf*, is any nine-tuple of arcs $f_{i,j}$, $i, j \in [3]$, such that their endpoints form a six-element set

$$K = \{\overline{p_1}, \overline{p_2}, \overline{p_3}, \overline{q_1}, \overline{q_2}, \overline{q_3}\} \subset \mathbb{R}^2,$$

for every pair $i, j \in [3]$ the arc $f_{i,j}$ joins $\overline{p_i}$ to $\overline{q_j}$, and the nine interiors $f_{i,j}^0$ are pairwise disjoint and disjoint to K . Graph-theoretically, a *$K_{3,3}$ -conf* is a plane drawing (i.e., without crossings) of the complete bipartite graph $K_{3,3}$.

Exercise 10 *Explain why no $K_{3,3}$ -conf exists. Hint: recall the course Discrete Mathematics.*

A PL *C-configuration*, abbreviated PL *C-conf*, is any *C-conf* in which the circuit f is a PL map. Similarly, a PL *$K_{3,3}$ -configuration*, abbreviated PL *$K_{3,3}$ -conf*, is any *$K_{3,3}$ -conf* in which all nine arcs $f_{i,j}$ are PL maps. Now we start the proof of Theorem 4.

• *Thomassen's proofs*. His proof has four steps.

1. $\exists C\text{-conf} \Rightarrow \exists K_{3,3}\text{-conf}$
2. $\exists K_{3,3}\text{-conf} \Rightarrow \exists \text{PL } K_{3,3}\text{-conf}$
3. $\exists \text{PL } K_{3,3}\text{-conf} \Rightarrow \exists \text{PL } C\text{-conf}$
4. $\exists \text{PL } C\text{-conf} \Rightarrow 0 = 1$

When we prove these four implications, Theorem 4 will follow. The main invention of Thomassen is the first step.

Exercise 11 *Exercise 10 says that $\exists K_{3,3}\text{-conf} \Rightarrow 0 = 1$. Does not this simplify our proof?*

• *The first step $\exists C\text{-conf} \Rightarrow \exists K_{3,3}\text{-conf}$. See the picture <https://kam.mff.cuni.cz/~klazar/JordanPic2.pdf>*

Let $f: I = I_{a,b} \rightarrow \mathbb{R}^2$ be a C -conf, i.e., f is a circuit such that the open set $\mathbb{R}^2 \setminus f[I]$ is connected. We enclose $f[I]$ in a rectangle $R \supset f[I]$ (Exercise 12) such that

$$\partial R \cap f[I] = \{\overline{p_1}, \overline{p_2}\}$$

where ∂R is the (rectangular) boundary of R and $\overline{p_1}$ (resp. $\overline{p_2}$) is an interior point of the bottom (resp. top) side of R . Let U be the part of ∂R between $\overline{p_1}$ and $\overline{p_2}$ containing the right side of R . We may assume that

$$a \leq t = f^{-1}(\overline{p_1}) < t' = f^{-1}(\overline{p_2}) \leq b$$

where at least one \leq is strict. We split f in two halves, the arcs

$$f_1 = f|_{I' = [t, t']} \quad \text{and} \quad f_2 = f|_{I'' = [t', t + b - a]}.$$

Let $S \subset R$ be any segment parallel to the bottom side of R and with endpoints in the interiors of the left and right sides of R . It follows (Exercise 13) that there exists a subsegment $T \subset S$ with endpoints $\overline{q_2} \in f_1[I']$ and $\overline{q_3} \in f_2[I'']$ and with interior disjoint to $f[I]$. From the assumption that $\mathbb{R}^2 \setminus f[I]$ is connected and from Theorem 6 it follows (Exercise 14) that there is a PL arc $f_{3,1}$ with image disjoint to $f[I]$, interior disjoint to $T \cup U$ and joining a point $\overline{p_3}$ in the interior of T to a point $\overline{q_1}$ in the interior of U .

We describe the nine arcs forming a $K_{3,3}$ -conf. The arc $f_{1,1}$ is the part of U from $\overline{p_1}$ to $\overline{q_1}$, $f_{1,2}$ is the initial part of f_1 from $\overline{p_1}$ to $\overline{q_2}$, $f_{1,3}$ is the reversed final part of f_2 from $\overline{q_3}$ to $\overline{p_1}$, $f_{2,1}$ is the the part of U from $\overline{p_2}$ to $\overline{q_1}$, $f_{2,2}$ is the reversed final part of f_1 from $\overline{q_2}$ to $\overline{p_2}$, $f_{2,3}$ is the initial part of f_2 from $\overline{p_2}$ to $\overline{q_3}$, $f_{3,1}$ was already defined above, and $f_{3,2}$ (resp. $f_{3,3}$) is the straight segment joining $\overline{p_3}$ to $\overline{q_2}$ (resp. $\overline{q_3}$). It follows from these definitions that the required disjointness conditions hold and we indeed have a $K_{3,3}$ -conf.

Exercise 12 *Explain how to find the rectangle R .*

Exercise 13 *Show that the subsegment T exists. Hint: intermediate values of continuous functions.*

Exercise 14 *Show that the arc $f_{3,1}$ exists.*

• *The second step $\exists K_{3,3}$ -conf $\Rightarrow \exists$ PL $K_{3,3}$ -conf. See <https://kam.mff.cuni.cz/~klazar/JordanPic3.pdf>*

Suppose that $f_{i,j}$, $i, j \in [3]$, is a $K_{3,3}$ -conf as obtained above, with the endpoints $\overline{p_i}$ and $\overline{q_j}$. Let $O_{i,j} \subset \mathbb{R}^2$ be the image of $f_{i,j}$ and $d > 0$ be the minimum of the distances between two of the six endpoints and between $O_{i,j}$ and an endpoint different from $\overline{p_i}$ and $\overline{q_j}$. We take the six closed discs

$$D(i) = \overline{B}(\overline{p_i}, d/3) \quad \text{and} \quad E(j) = \overline{B}(\overline{q_j}, d/3), \quad i, j \in [3].$$

Any two discs have distance $\geq d/3$. It follows (Exercise 15) that for every $i, j \in [3]$ there exists the last time $t_{i,j} \in \mathbb{R}$ when $f_{i,j}$ exits $D(i)$ and the first following time $u_{i,j} > t_{i,j}$ when $f_{i,j}$ enters $E(j)$. It follows that for every $k, l \in [3]$,

$$f_{i,j}[(t_{i,j}, u_{i,j})] \cap (D(k) \cup E(l)) = \emptyset.$$

The exit and entrance points (which lie on the boundaries of $D(i)$ and $E(j)$, respectively) are

$$\overline{l_{i,j}} = f_{i,j}(t_{i,j}) \in \partial D(i) \quad \text{and} \quad \overline{e_{i,j}} = f_{i,j}(u_{i,j}) \in \partial E(j) .$$

For $i, j \in [3]$ we define the arc

$$f_{i,j}^{(1)} : J_{i,j} = [t'_{i,j}, u'_{i,j}] \rightarrow \mathbb{R}^2, \quad \text{for some } t'_{i,j} < t_{i,j} \text{ and } u'_{i,j} > u_{i,j} ,$$

so that on $[t'_{i,j}, t_{i,j}]$ the arc $f_{i,j}^{(1)}$ is the segment $s(\overline{p_i}, \overline{l_{i,j}})$, on $I_{i,j} = [t_{i,j}, u_{i,j}]$ it coincides with $f_{i,j}$ and on $[u_{i,j}, u'_{i,j}]$ it is the segment $s(\overline{e_{i,j}}, \overline{q_j})$. We consider the minimum distance

$$e = \min \left(\{e_2(f_{i,j}[I_{i,j}], f_{k,l}^{(1)}[J_{k,l}]) \mid i, j, k, l \in [3], (i, j) \neq (k, l)\} \right) > 0$$

(Exercise 16). The restricted arcs $f_{i,j} : I_{i,j} \rightarrow \mathbb{R}^2$ are uniformly continuous (Exercise 17) and therefore $\exists \delta > 0$ such that for every $i, j \in [3]$ and every $t, u \in I_{i,j}$,

$$|t - u| \leq \delta \Rightarrow e_2(f_{i,j}(t) - f_{i,j}(u)) \leq \min(\{e/6, d/6\}) .$$

For any $i, j \in [3]$ we take a partition $t_{i,j} = v_0 < v_1 < \dots < v_n = u_{i,j}$ of $I_{i,j}$ (its dependence on i, j is not marked) such that $v_k - v_{k-1} \leq \delta$ and define

$$f_{i,j}^{(2)} : J_{i,j} \rightarrow \mathbb{R}^2$$

as $f_{i,j}^{(2)} = f_{i,j}^{(1)}$ on $[t'_{i,j}, t_{i,j}] \cup [u_{i,j}, u'_{i,j}]$ and as the PL map with the segments $s(f_{i,j}(v_{r-1}), f_{i,j}(v_r))$, $r = 1, 2, \dots, n$, on $[t_{i,j}, u_{i,j}]$. It follows from the choice of δ that the interiors $f_{i,j}^{(2)}[I_{i,j}]^0$ are pairwise disjoint, because for every $i, j, k, l \in [3]$ with $(i, j) \neq (k, l)$ one has that

$$e_2(f_{i,j}^{(2)}[I_{i,j}], f_{k,l}^{(2)}[J_{k,l}]) \geq e/3 ,$$

and that they are also disjoint to all six endpoints (Exercise 18). Finally, using Proposition 9 we replace the PL maps $f_{i,j}^{(2)}$ with the PL arcs

$$f_{i,j}^{(3)}: J_{i,j} \rightarrow \mathbb{R}^2$$

such that $f_{i,j}^{(3)}[J_{i,j}] \subset f_{i,j}^{(2)}[J_{i,j}]$ and that $f_{i,j}^{(3)}$ joins \bar{p}_i to \bar{q}_j . It follows that the PL arcs $f_{i,j}^{(3)}$, $i, j \in [3]$, form a PL $K_{3,3}$ configuration.

Exercise 15 *Prove that the exit and entrance times for the arcs $f_{i,j}$ with respect to the discs $D(i)$ and $E(j)$ exist.*

Exercise 16 *Prove that the distance e is positive.*

Exercise 17 *Why are the restricted arcs $f_{i,j}: I_{i,j} \rightarrow \mathbb{R}^2$ uniformly continuous?*

Exercise 18 *Explain why are the interiors $f_{i,j}^{(2)}[I_{i,j}]^0$ pairwise disjoint and disjoint to the six endpoints.*

• *The third step \exists PL $K_{3,3}$ -conf $\Rightarrow \exists$ PL C -conf. See <https://kam.mff.cuni.cz/~klazar/JordanPic4.pdf>*

We show that any PL $K_{3,3}$ -conf contains as a subgraph a PL C -conf.

Let f be an oriented PL circuit. Each segment s of f then determines the *right open halfplane* $\text{rp}(s) \subset \mathbb{R}^2$ of points in \mathbb{R}^2 lying to the right of the line extending s . We similarly define the *left open halfplane* $\text{lp}(s) \subset \mathbb{R}^2$. For $n \in \mathbb{N}$ the *right shadow* $r(s, n)$ of s is the segment $s' \subset \text{rp}(s)$ whose endpoints are the two points in $\text{rp}(s)$ that lie on the two axes of the two endpoints (corners) of s in distance $1/n$ from the endpoint of s . We define the *left shadow* $l(s, n)$, $n \in \mathbb{N}$, of s in the same way, only $\text{rp}(s)$ is replaced with

$lp(s)$. For $n \in \mathbb{N}$ we define the *right shadow* $r(f, n)$ and the *left shadow* $l(f, n)$ of the oriented PL circuit f by

$$r(f, n) = \bigcup_{s \in S(f)} r(s, n) \quad \text{and} \quad l(f, n) = \bigcup_{s \in S(f)} l(s, n)$$

where $S(f)$ is the set of segments of f .

Proposition 19 (on shadows 1) \forall oriented PL circuit f and $\forall n$, both shadows $r(f, n)$ and $l(f, n)$ are images of PL maps.

Proof. This is immediate from their definitions. \square

Proposition 20 (on shadows 2) Let $f: I \rightarrow \mathbb{R}^2$ be an oriented PL circuit. There is an n_0 such that for every $n \geq n_0$,

$$r(f, n) \cap f[I] = \emptyset = l(f, n) \cap f[I] .$$

Proof. Let f be as stated and $d = \min_{s, s'} e_2(s, s') > 0$ where s, s' run through all pairs of segments of f with $s \cap s' = \emptyset$. Let s be any segment of f . It suffices to prove that for n large enough, $r(s, n) \cap f[I] = \emptyset$; for $l(s, n)$ the arguments is similar. Let s' and s'' be the two segments of f adjacent to s . It is easy to see that $r(s, n) \cap s = \emptyset$ for every n and that $r(s, n) \cap (s' \cup s'') = \emptyset$ for every large n (Exercise 21). Also,

$$r(s, n) \subset \{\bar{p} \in \mathbb{R}^2 \mid e_2(\{\bar{p}\}, s) \leq 1/n\} .$$

Thus it suffices to take n so large that $r(s, n) \cap (s' \cup s'') = \emptyset$ and that $1/n \leq d/3$. \square

Exercise 21 Show that for $n \geq n_0$, neither $r(s, n)$ nor $l(s, n)$ intersects the two segments of the PL circuit adjacent to s .

Proposition 22 (on shadows 3) *Let $f: I \rightarrow \mathbb{R}^2$ be an oriented PL circuit. Then for any point $\bar{p} \in \mathbb{R}^2 \setminus f[I]$ one of two cases occurs.*

(L) *For every $n \geq n_0$ a PL arc in $\mathbb{R}^2 \setminus f[I]$ joins the point \bar{p} to a point in $l(f, n)$.*

(R) *For every $n \geq n_0$ a PL arc in $\mathbb{R}^2 \setminus f[I]$ joins the point \bar{p} to a point in $r(f, n)$.*

Proof. Let f and \bar{p} be as stated and let u be any segment realizing the distance between \bar{p} and $f[I]$. Then \bar{p} is one endpoint of u , the other one $\bar{q} \in f[I]$ and $u^0 \subset \mathbb{R}^2 \setminus f[I]$. Considering u near \bar{q} , we see that (L) or (R) occurs. \square

For an oriented PL circuit f we define $A_{f,R} \subset \mathbb{R}^2$ (resp. $A_{f,L} \subset \mathbb{R}^2$) as those points \bar{p} in the complement of the image of f for which the above case (R) (resp. (L)) holds.

Corollary 23 (left and right sides) *Let $f: I \rightarrow \mathbb{R}^2$ be an oriented PL circuit. Then*

$$\mathbb{R}^2 \setminus f[I] = A_{f,R} \cup A_{f,L}$$

and $A_{f,R}$ and $A_{f,L}$ are connected open sets.

Proof. It is clear that $A_{f,R}$ and $A_{f,L}$ are open sets. Indeed, let $\bar{p} \in A_{f,R}$, say, witnessed by a PL arc g joining \bar{p} in the complement of $f[I]$ to a point in $r(f, n)$. Let the ball $B = B(\bar{p}, r)$ have radius $r > 0$ so small that $B \subset \mathbb{R}^2 \setminus f[I]$ and that B intersect only one segment of g . Then for every $\bar{q} \in B$ we can easily modify g to a PL arc joining \bar{q} in the complement of $f[I]$ to the same point in $r(f, n)$.

Let $\bar{p}, \bar{q} \in A_{f,R}$ be two distinct points (for $A_{f,L}$ the argument is similar). We show that there is a PL arc g that joins \bar{p} to \bar{q} in $\mathbb{R}^2 \setminus f[I]$. Then $A_{f,R}$ is connected by Theorem 7. We use Propositions 20 and 22 and the definition of the set $A_{f,R}$ and take large enough n such that $r(f, n) \cap f[I] = \emptyset$ and that there are PL arcs joining, respectively, \bar{p} and \bar{q} in $\mathbb{R}^2 \setminus f[I]$ to points in $r(f, n)$. By Propositions 9 and 19 there exists the required PL arc g . \square

Now suppose that the nine arcs $f_{i,j}$, $i, j \in [3]$, form a PL $K_{3,3}$ -conf. Let $k: I \rightarrow \mathbb{R}^2$ be the oriented PL circuit formed by the six arcs $f_{1,1}$, $f_{2,1}$, $f_{2,2}$, $f_{3,2}$, $f_{3,3}$ and $f_{1,3}$. We denote the remaining three arcs by $e = f_{1,2}$, $g = f_{2,3}$ and $h = f_{3,1}$. We write $\mathbb{R}^2 \setminus k[I] = A_{k,R} \cup A_{k,L}$ as in Corollary 23. If $A_{k,R}$ and $A_{k,L}$ intersect then k is a PL C -conf and we are done. Hence these sets are disjoint. Then the interior of each of the arcs e , g and h lies completely in $A_{k,R}$ or completely in $A_{k,L}$ (else $A_{k,R}$ and $A_{k,L}$ would cut the interior of the arc, which is however a connected set). Thus two of these interiors lie in the same set, for example (other cases are similar) the interiors of e and h lie in $A_{k,R}$. We consider the oriented PL circuit $l: I' \rightarrow \mathbb{R}^2$ formed by the arcs $f_{2,2}$, $f_{2,1}$, $f_{1,1}$ and e ; we orient the segments in e consistently with those in the other three PL arcs. But we see that the interior h^0 of h intersects both $A_{l,R}$ and $A_{l,L}$ (Exercise 24). By Corollary 23,

$$\mathbb{R}^2 \setminus l[I'] = A_{l,R} \cup h^0 \cup A_{l,L}$$

is connected and l is a PL C -conf (Exercise 25).

Exercise 24 *Why does h^0 intersect both the right and the left side of the oriented PL circuit l ?*

Exercise 25 Why is the set $A_{l,R} \cup h^0 \cup A_{l,L}$ connected?

• The fourth step \exists PL C -conf $\Rightarrow 0 = 1$. See the picture <https://kam.mff.cuni.cz/~klazar/JordanPic5.pdf>

We suppose that $f: I \rightarrow \mathbb{R}^2$ is a PL circuit with connected complement $\mathbb{R}^2 \setminus f[I]$ and deduce a contradiction. It easily follows from the next proposition.

Proposition 26 (in or out?) Let $f: I \rightarrow \mathbb{R}^2$ be a PL circuit and $D = \mathbb{R}^2 \setminus f[I]$. There exists a continuous map

$$g: D \rightarrow \{0, 1\} \subset \mathbb{R}$$

such that $g[D] = \{0, 1\}$.

Proof. Let f and D be as stated. We may assume (Exercise 28) that none of the finitely many segments of $f[I]$ is vertical. For any point $\bar{p} = (p_x, p_y) \in D$ we denote by

$$r(\bar{p}) = (x = p_x, y \geq p_y) \subset \mathbb{R}^2$$

the vertical ray (half-line) emanating upwards from \bar{p} . We define the function $g: D \rightarrow \{0, 1\}$ as the parity of the finite sum (Exercise 29)

$$g(\bar{p}) = \sum_{\bar{q} \in r(\bar{p}) \cap f[I]} m(\bar{q}) \pmod{2}$$

(for empty intersection the sum is 0), where the multiplicity $m(\bar{q}) \in \{0, 1\}$ of the intersection \bar{q} of the ray $r(\bar{p})$ and the graph $f[I]$ of the circuit f is defined as follows. If \bar{q} is transversal, meaning that $f[I]$ lies locally near \bar{q} on both sides of $r(\bar{p})$, we set $m(\bar{q}) = 1$. Else, when $f[I]$ lies locally near \bar{q} only on one side of $r(\bar{p})$, we set $m(\bar{q}) = 0$. It follows that $m(\bar{q}) = 0$ iff \bar{q} is the common corner of two consecutive segments of $f[I]$ lying on the same side of $r(\bar{p})$ (Exercise 30).

We prove that g is continuous. Let $\bar{p} = (p_x, p_y) \in D$ be an arbitrary point and $X \subset \mathbb{R}$ be the finite set of the x -coordinates of all corners of f , minus the number $\{p_x\}$. Let

$$\delta = \min \left(\{|p_x - a| \mid a \in X\} \cup \{e_2(\{\bar{p}\}, f[I])\} \right) > 0 .$$

We claim that

$$\bar{q} \in B(\bar{p}, \delta) \Rightarrow g(\bar{q}) = g(\bar{p}) .$$

To see it, compare for such \bar{q} the finite intersections $X_1 = r(\bar{p}) \cap f[I]$ and $X_2 = r(\bar{q}) \cap f[I]$. By the choice of δ , when we move the point \bar{p} to the position \bar{q} every transversal intersection $\bar{r} \in X_1$ transforms in a transversal intersection $\bar{r}' \in X_2$, and every non-transversal intersection $\bar{r} \in X_1$ either disappears or remains the same non-transversal intersection $\bar{r} \in X_2$ or transforms in two distinct transversal intersections $\{\bar{r}', \bar{r}''\} \subset X_2$. Also, all intersections in X_2 arise in these ways, no new intersection can appear. Thus $g(\bar{q}) = g(\bar{p})$ by the definition of g .

It remains to show that $g[D] = \{0, 1\}$. Let $\bar{q} \in f[I]$ be one of the highest corners of $f[I]$, i.e., with the maximum y -coordinate, and let ℓ be the axis of \bar{q} . Then it is easy to see that $g(\bar{p}) = 0$ for every point $\bar{p} \in \ell$ lying above \bar{q} and that $g(\bar{p}) = 1$ for every point $\bar{p} \in \ell$ lying below \bar{q} and sufficiently close to \bar{q} . \square

To obtain a contradiction for a PL circuit f possessing connected complement $D = \mathbb{R}^2 \setminus f[I]$, we take the function $g: D \rightarrow \{0, 1\}$ guaranteed by the previous proposition and take two points \bar{p} and \bar{q} in D such that $g(\bar{p}) = 0$ and $g(\bar{q}) = 1$. By Theorem 6 there is a PL arc $h: J \rightarrow \mathbb{R}^2$ with $h[J] \subset D$ and joining \bar{p} to \bar{q} . But then $g(h): J \rightarrow \{0, 1\}$ is a continuous function that maps the connected

interval J to the disconnected set $g(h)[J] = \{0, 1\} \subset \mathbb{R}$, which is indeed a contradiction.

This completes the proof of Theorem 4.

Exercise 27 *Where do we run in difficulties when we attempt to define the function g in the same way for a PL arc? By Theorem 3, it has to fail somewhere.*

Exercise 28 *Why can we assume that none of the segments of the PL circuit f is vertical?*

Exercise 29 *Why is the displayed sum in the proof finite?*

Exercise 30 *Prove the equivalence characterizing geometrically the intersection points \bar{q} with multiplicity zero.*

• *Concluding and other remarks.* The full Jordan theorem, which Jordan basically correctly proved (as discussed by Hales at <http://mizar.org/trybulec65/4.pdf>) in the book *Course d'analyse de l'École Polytechnique*, Paris, 1893, is as follows.

Theorem 31 (the Full Jordan Theorem) *For every plane circuit $f: I \rightarrow \mathbb{R}^2$ it holds that*

$$\mathbb{R}^2 \setminus f[I] = A_{\text{int}} \cup A_{\text{ext}}$$

where A_{int} and A_{ext} are nonempty open connected sets that are disjoint. Moreover, A_{int} is bounded and A_{ext} is unbounded.

Exercise 32 *How does the fact that $\mathbb{R}^2 \setminus f[I]$ is disconnected (i.e., Theorem 4 we have just proven) follow from the theorem?*

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 1, 5, 13, 16 and 27.