MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2024/25
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LECTURE 4 (March 12, 2025) THE PROOF OF FUNDAMENTAL THEOREM OF ALGEBRA. COMPLETE SPACES. BAIRE'S THEOREM

• *n-th complex roots.* To prove the existence of n-th roots in $\mathbb{C}$, we first reduce the situation to odd n and to numbers with modulus 1 which lie on the complex unit circle S.

Exercise 1 *Using the last two exercises in the previous lecture, prove that if for every $u \in S$ and for every odd $n \in \mathbb{N}$ there exists a $v \in S$ such that $v^n = u$, then the following theorem holds.*

Theorem 2 (n-th roots in $\mathbb{C}$) *Complex numbers contain all n-th roots, formally*

$$\forall\, u \in \mathbb{C}\; \forall\, n \in \mathbb{N}\; \exists\, v \in \mathbb{C}\, \big(v^n = u\big)\ .$$

Proof. So we can assume that $u \in S$ and $n \in \mathbb{N}$ is odd. We need to prove that the map

$$f(z) = z^n\colon S \to S\ ,$$

which is clearly continuous, is onto. We assume for contradiction that there is a number

$$w \in S \setminus f[S]$$

(that is, w has no n-th root). Since n is odd, also $-w \in S \setminus f[S]$ (always $f(-z) = -f(z)$). We consider the line $\ell \subset \mathbb{C}$ going through the points w and $-w$. Then we have the partition

$$\mathbb{C} = A \cup \ell \cup B \ ,$$

where A and B are open half-planes determined by ℓ. By Exercise 3, A and B are disjoint open sets. By Exercise 4, $(A \cup B) \cap S = S \setminus \{w, -w\}$, $\{1, -1\} \subset f[S] \cap (A \cup B)$ and $|A \cap \{1, -1\}| = 1$. Thus, the sets A and B cut the set $f[S]$ and make it disconnected. This contradicts Theorem 21 in the last lecture, because $f[S]$ is the image of the connected set S by the continuous function f and is therefore connected. □

Exercise 3 *Prove that for every line $\ell \subset \mathbb{C}$, $\mathbb{C} \setminus \ell$ is the disjoint union of two open sets.*

Exercise 4 *Let $\ell \subset \mathbb{C}$ be a line passing through the origin, $\ell \cap S = \{w, -w\}$ and A and B are the open half-planes determined by it. Prove that $(A \cup B) \cap S = S \setminus \{w, -w\}$ and that for every $u \in S \setminus \{w, -w\}$, the points u and $-u$ lie in different half-planes A and B.*

We proceed to the second step of the proof of FTAlg which uses compact sets in $\mathbb{C}$. Recall that the complex numbers $\mathbb{C}$ form the MS $(\mathbb{C}, |u - v|)$ which is isometric to the Euclidean plane $(\mathbb{R}^2, e_2)$.

Exercise 5 *For every real numbers $\alpha \le \alpha'$ and $\beta \le \beta'$, the rectangle*

$$R = \{a + bi\colon\ \alpha \le a \le \alpha' \wedge \beta \le b \le \beta'\}$$

is a compact set.

Proposition 6 (reduction to n-th roots) *If $\mathbb{C}$ contains all roots, then the Fundamental Theorem of Algebra holds, every non-constant complex polynomial has a root.*

Proof. Let

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

be a non-constant complex polynomial: $n \in \mathbb{N}$, $a_j \in \mathbb{C}$ and $a_n \neq 0$. The function

$$f(z) = |p(z)| \colon \mathbb{C} \to [0, +\infty) \subset \mathbb{C}$$

is continuous. We prove that $f(u) = 0$ for some $u \in \mathbb{C}$. Then also $p(u) = 0$ and u is a root of $p(z)$.

First we prove that f attains on its definition domain $\mathbb{C}$ a minimum value $f(u)$. Then we prove that $f(u) = 0$. Let the real number $K > 0$ be so large that

$$\frac{K^n |a_n|}{2} > |a_0| \ \text{ and } \sum_{j=0}^{n-1} |a_j| K^{j-n} < \frac{|a_n|}{2} \ .$$

Then for every $z \in \mathbb{C}$ we have the estimate that

$$\begin{aligned} |z| > K \Rightarrow f(z) = |p(z)| \ &\geq \ |z|^n \bigg(|a_n| - \sum_{j=0}^{n-1} |a_j| \cdot |z|^{j-n} \bigg) \\ &> \ |a_0| = |p(0)| = f(0) \ . \end{aligned}$$

We define a rectangle

$$R = \{a + bi \colon \ -K \leq a,\, b \leq K\} \subset \mathbb{C} \ .$$

Clearly, if $z \in \mathbb{C} \setminus R$ then $|z| > K$. By Theorem 15 in the second lecture (the minimax principle) and Exercise 5 in this lecture there exists $u \in R$ such that $f(u) \leq f(v)$ for every $v \in R$. Since $0 \in R$, $f(u) \leq f(0)$. By the above estimate we have that

$$\forall\, v \in \mathbb{C} \left(f(u) \leq f(v) \right) .$$

Thus f attains at u the smallest value on $\mathbb{C}$.

We prove that $f(u) = 0$. To this end we express the polynomial $p(z)$ by Exercise 7 as

$$p(z) = \sum_{j=0}^{n} b_j (z-u)^j ,$$

with $b_j \in \mathbb{C}$ and $b_n = a_n$. So $f(u) = |p(u)| = |b_0|$. Let for contrary $f(u) = |b_0| > 0$. We find the first non-zero non-constant coefficient b_k in $p(z)$ and write it as

$$p(z) = b_0 + b_k(z-u)^k + \underbrace{b_{k+1}(z-u)^{k+1} + \cdots + b_n(z-u)^n}_{q(z)} ,$$

where $q \in \mathbb{C}[z]$, $k \in \mathbb{N}$, $b_0 \neq 0$ and $b_k \neq 0$. We use the assumption of existence of roots in $\mathbb{C}$ and take an $\alpha \in \mathbb{C}$ such that

$$\alpha^k = -\frac{b_0}{b_k} .$$

It is clear that $q(z) = o((z-u)^k)$ (for $z \to u$), so that

$$\lim_{z \to u} q(z)(z-u)^{-k} = 0 .$$

Hence we can take a $\delta \in (0,1)$ such that for

$$v = u + \delta\alpha$$

one has

$$|q(v)| < \delta^k \cdot \frac{|b_0|}{2} .$$

We get the contradiction that $f(v) < f(u)$:

$$\begin{aligned} f(v) = |p(v)| &= |b_0 + b_k\alpha^k\delta^k + q(v)| \\ &\overset{\text{def. of } \alpha}{=} |b_0(1-\delta^k) + q(v)| \\ &\overset{\Delta\text{'s ineq. and mult. } |\cdot|}{\leq} |b_0|(1-\delta^k) + |q(v)| \\ &\overset{|q(v)| < \dots}{<} |b_0|(1-\delta^k/2) \\ &\overset{\delta \in (0,\,1)}{<} |b_0| = f(u)\ . \end{aligned}$$

Hence $f(u) = 0$ and $p(u) = 0$. □

Exercise 7 *Prove that for eny $n \in \mathbb{N}_0$ and any complex numbers $a_0,\ a_1,\ \dots,\ a_n$ and u there exist complex numbers $b_0,\ b_1,\ \dots,\ b_n$ such that $b_n = a_n$ and the polynomial equality*

$$\sum_{j=0}^{n} a_j z^j = \sum_{j=0}^{n} b_j (z-u)^j$$

holds.

• *Complete metric spaces and complete sets.* Let (M, d) be a MS. It is *complete* if every Cauchy sequence $(a_n) \subset M$ is convergent. Recall that *Cauchy sequence* (a_n) satisfies

$$\forall\, \varepsilon\ \exists\, n_0\, \big(m,\ n \geq n_0 \Rightarrow d(a_m,\ a_n) < \varepsilon\big)\ .$$

A set $X \subset M$ is *complete* if the subspace (X, d) is complete.

Exercise 8 *Let (M, d) be* MS *and $X \subset Y \subset M$. Prove that a set X is complete in the* MS (Y, d) *if and only if it is complete in the* MS (M, d)*.*

Exercise 9 *Prove that the Cartesian product*

$$(M \times N,\, d \times e)$$

of complete MS*s* (M, d) *and* (N, e) *is a complete* MS*.*

A basic example of a complete MS is the Euclidean space

$$(\mathbb{R},\, e_1) = (\mathbb{R},\, |x - y|)\ .$$

It is complete due to the fact that every sequence $(a_n) \subset \mathbb{R}$ is convergent if and only if it is Cauchy. By Exercise 9 all Euclidean spaces $(\mathbb{R}^n, e_n)$, $n \in \mathbb{N}$, are complete. We can construct many complete MSs as follows.

Proposition 10 (closed subspaces) *In every complete* MS (M, d) *every closed subset* $X \subset M$ *is complete.*

Proof. Let $(a_n) \subset X$ be a Cauchy sequence in the closed set $X \subset M$ in the complete MS (M, d). There exists $a = \lim a_n \in M$. Since X is a closed set, $a \in X$. So the set X is complete. □

Exercise 11 *Let* $X \subset M$ *be a compact set in a* MS (M, d)*. Prove that* X *is complete.*

Exercise 12 *Give an example of a complete and non-compact set* $X \subset \mathbb{R}$ *in the Euclidean* MS $(\mathbb{R}, e_1)$*.*

Exercise 13 *Which of the following implications holds in a* MS (M, d)*?*

1. $X \subset M$ *is a complete set* $\Rightarrow$ X *is closed.*
2. $X \subset M$ *and* $Y \subset M$ *are complete sets* $\Rightarrow$ $X \cup Y$ *is a complete set.*

3. *$X \subset M$ and $Y \subset M$ are complete sets $\Rightarrow$ $X \cap Y$ is a complete set.*

4. *$X \subset M$ is a complete set $\Rightarrow$ X is bounded.*

5. *$X \subset M$ is finite $\Rightarrow$ X is complete.*

• *Baire's theorem.* This is the main result about complete metric spaces. It says that no complete MS is a countable union of sparse sets. A set $X \subset M$ in a MS (M, d) is *sparse (in M)* if

$$\forall\, a \in M\ \forall\, r > 0\ \exists\, b \in M\ \exists\, s > 0$$
$$\big(B(b,\, s) \subset B(a,\, r) \wedge B(b,\, s) \cap X = \emptyset\big)$$

– every ball in (M, d) contains a subball disjoint to X. Similarly, a set $X \subset M$ in a MS (M, d) is *dense (in M)* if

$$\forall\, a \in M\ \forall\, r > 0\, \big(B(a,\, r) \cap X \neq \emptyset\big)$$

– every ball in (M, d) contains an element of the set X.

Exercise 14 *Let (M, d) be a* MS *and $X \subset M$ be a subset. Prove the equivalence that*

$$X \text{ is dense} \iff \forall\, a \in M\ \exists\, (a_n) \subset X\, \big(\lim a_n = a\big)\ .$$

Proposition 15 (density and continuity) *Let (M, d) and (N, e) be* MS*s, $X \subset M$ be dense in M and let*

$$f,\, g\colon M \to N$$

be continuous mappings such that $f \,|\, X = g \,|\, X$ (their restrictions to X coincide). Then $f = g$.

Proof. Let $a \in M$ be an arbitrary point. Since X is dense, by the previous exercise there exists a sequence $(a_n) \subset X$ such that $\lim a_n = a$. Using Heine's definition of continuity of functions and the assumption about f and g, we have that

$$f(a) = \lim f(a_n) = \lim g(a_n) = g(a).$$

So $f = g$. □

Exercise 16 *Any finite union of sparse sets is a sparse set. This is in general not true for countable unions.*

Exercise 17 *The intersection of two dense sets, one of which is open, is a dense set. This is in general not true if we omit the assumption of openness.*

For $a \in M$ and real $r > 0$, the *closed ball* $\overline{B}(a, r)$ in a MS (M, d) is the set

$$\overline{B}(a,\, r) = \{x \in M \mid d(a,\, x) \le r\}\ .$$

Exercise 18 *Every closed ball $\overline{B}(a, r)$ is a closed set. For every $a \in M$ and $r, s \in \mathbb{R}$ with $0 < r < s$,*

$$\overline{B}(a,\, r) \subset B(a,\, s)\ .$$

Theorem 19 (Baire's) *Let (M, d) be a complete* MS *and*

$$M = \bigcup_{n=1}^{\infty} X_n\ .$$

Then for some n, the set X_n is not sparse. In other words, no complete metric space is a countable union of sparse sets.

Proof. We assume that all sets X_n are sparse and deduce a contradiction. We construct a sequence $(\overline{B_n})$ of nested closed balls with centers converging to a point $a \in M$ not in any X_n, which is a contradiction.

Let $B(b, 1) \subset M$ be arbitrary. X_1 is sparse and there exists an a_1 in M and an $s_1 > 0$ such that $B(a_1, s_1) \subset B(b, 1)$ and $B(a_1, s_1) \cap X_1 = \emptyset$. We set

$$\overline{B}(a_1,\, r_1) = \overline{B}(a_1,\, \min(s_1/2,\, 1/2)) \ .$$

Then $\overline{B}(a_1, r_1) \subset B(a_1, s_1)$, thus $\overline{B}(a_1, r_1) \cap X_1 = \emptyset$, and $r_1 \leq 1/2$.

Suppose that we already defined the closed balls

$$\overline{B}(a_1,\, r_1) \supset \overline{B}(a_2,\, r_2) \supset \cdots \supset \overline{B}(a_n,\, r_n)$$

such that for $i = 1, 2, \ldots, n$, $\overline{B}(a_i, r_i) \cap X_i = \emptyset$ and $r_i \leq 2^{-i}$. Since X_{n+1} is sparse, there exist $a_{n+1} \in M$ and $s_{n+1} > 0$ such that $B(a_{n+1}, s_{n+1}) \subset B(a_n, r_n)$ and $B(a_{n+1}, s_{n+1}) \cap X_{n+1} = \emptyset$. We set

$$\overline{B}(a_{n+1},\, r_{n+1}) = \overline{B}(a_{n+1},\, \min(s_{n+1}/2,\, 2^{-n-1})) \ .$$

Then

$$\overline{B}(a_{n+1},\, r_{n+1}) \subset \overline{B}(a_n,\, r_n) \cap B(a_{n+1},\, s_{n+1}) \ .$$

Hence also $\overline{B}(a_{n+1}, s_{n+1}) \cap X_{n+1} = \emptyset$ and $r_{n+1} \leq 2^{-n-1}$.

The sequence $(a_n) \subset M$ of the centers of the closed balls defined above is Cauchy:

$$m \geq n \Rightarrow \overline{B}(a_m,\, r_m) \subset \overline{B}(a_n,\, r_n) \ \text{ and } \ d(a_m,\, a_n) \leq r_n \leq \frac{1}{2^n} \ .$$

We use completeness of the MS (M, d) and take the limit

$$a = \lim a_n \in M \ .$$

Since $m \geq n \Rightarrow a_m \in \overline{B}(a_n, r_n)$ and since by Exercise 18 every $\overline{B}(a_n, r_n)$ is a closed set, the limit a lies in every closed ball $\overline{B}(a_n, r_n)$ and therefore in none of the sets X_n, which is a contradiction. $\square$

Baire's theorem has many applications, but now we mention only one. A point $a \in M$ in a MS (M, d) is *isolated* if

$$\exists\, r > 0 \left(B(a,\, r) = \{a\}\right) .$$

Exercise 20 *Prove that in any* MS (M, d),

$$a \in M \text{ is not isolated} \iff \{a\} \text{ is a sparse set in } M \ .$$

Corollary 21 (uncountability) *Any complete* MS (M, d) *without isolated points is uncountable.*

Proof. Suppose for the contrary that M is countable. Then

$$M = \bigcup_{a \in M} \{a\}$$

is a countable union. Since each set $\{a\}$ is sparse (by the previous exercise), we have a contradiction with Baire's theorem. $\square$

THANK YOU FOR YOUR ATTENTION

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 7, 11, 13, 16 and 20.