MATHEMATICAL ANALYSIS 3 (NMAI056)
summer term 2024/25
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LECTURE 12 (May 7, 2025) NEWMAN'S PROOF OF THE PRIME NUMBER THEOREM

• *The Prime Number Theorem*, abbreviated PNT, is the asymptotic estimate

$$\pi(x) \sim x(\log x)^{-1} \;\; (x \to +\infty)$$

of the prime number counting function $\pi(x)$, defined for any $x \in \mathbb{R}$ as the number of primes p such that $p \le x$. For example, $\pi(11.8) = |\{2, 3, 5, 7, 11\}| = 5$ and $\pi(x) = 0$ for every $x < 2$. In other words,

$$\lim_{x \to +\infty} \frac{\pi(x)}{x/\log x} = 1\,.$$

• *History.* PNT was conjectured around 1800 by *Carl Friedrich Gauss (1777–1855)*. It was proved in 1896 by *Jacques Hadamard (1865–1963)* and, in parallel, *Charles J. de la Vallée Poussin (1866–1962)*. In 1980 *Donald J. Newman (1930–2007)* discovered substantial simplifications in analytic proofs of PNT. His proof is the topic of this lecture. I follow the article

> D. Zagier, Newman's short proof of the Prime Number Theorem, *Amer. Mathem. Monthly* **104** (1997), 705–708,

and my lecture notes

> Analytic and Combinatorial Number Theory I, *KAM-DIMATIA Series*, preprint no. 968 (2010), v+92 pp.

• *Equivalence of PNT to $\vartheta(x) \sim x$ $(x \to +\infty)$.* We define the function $\vartheta(x) = \sum_{p \le x} \log p$ for $x \in \mathbb{R}$.

Proposition 1 (restating PNT) *It is true that*

$$PNT \iff \vartheta(x) \sim x \quad (x \to +\infty)\,.$$

Proof. Clearly, $\vartheta(x) = \sum_{p \le x} \log p \le \pi(x) \log x$. Also, for any $\varepsilon > 0$ we have

$$\vartheta(x) \ge \textstyle\sum_{x^{1-\varepsilon} < p \le x} \log p \ge (\pi(x) - x^{1-\varepsilon})(1-\varepsilon) \log x\,.$$

The equivalence follows from these two bounds. □

• *Čebyšev's bound.* Around 1852 *Pafnutij L. Čebyšev (1821–1894)* proved the weak form of PNT that

$$\vartheta(x) = \Theta(x) \quad (x \ge 2)$$

– $c_1 x \le \vartheta(x) \le c_2 x$ for every $x \ge 2$ and constants $c_i > 0$. We make use of the upper bound.

Proposition 2 ($\vartheta(x) = O(x)$) *We have*

$$\vartheta(x) = O(x) \quad (x \ge 2)$$

– $0 < \vartheta(x) \le cx$ *for every* $x \ge 2$ *and a constant* $c > 0$.

Proof. For any $n \in \mathbb{N}$,

$$\exp(\vartheta(2n) - \vartheta(n)) = \textstyle\prod_{n < p \le 2n} p \le \frac{(2n)!}{n! \cdot n!} = \binom{2n}{n} \le (1+1)^{2n} = 4^n\,.$$

Hence $\vartheta(2n) - \vartheta(n) \le (\log 4)n$. For $x \ge 2$ let $k \in \mathbb{N}$ be such that $2^{k-1} \le x < 2^k$. Then

$$\vartheta(x) \le \textstyle\sum_{j=1}^{k} (\vartheta(2^j) - \vartheta(2^{j-1})) \le (\log 4) \sum_{j=1}^{k} 2^{j-1} \le (2 \log 4) x\,.$$

□

• *Morera's theorem.* The following interesting theorem is due to the Italian engineer and mathematician *Giacinto Morera (1856–1909)*. Recall that $U \subset \mathbb{C}$ is an open set.

Theorem 3 (Morera) *Let $f\colon U \to \mathbb{C}$ be continuous and such that $\int_{\partial R} f = 0$ for every rectangle $R \subset U$. Then f is holomorphic.*

Proof.

□

Corollary 4 (holomorphic limits) *Let $f_n\colon U \to \mathbb{C}$, $n \in \mathbb{N}$, be a sequence of holomorphic functions with pointwise limit*

$$\lim f_n(z) = f(z) \quad (\colon U \to \mathbb{C})\,.$$

If the convergence is uniform on every compact subset of U, then f is holomorphic.

Proof. It follows from Morera's theorem − the uniform limit f is continuous and for any rectangle $R \subset U$ we have

$$\int_{\partial R} f = \int_{\partial R} \lim f_n = \lim \int_{\partial R} f_n = \lim 0 = 0\,.$$

□

Corollary 5 (removable singularity) *If $f\colon U \to \mathbb{C}$ is continuous, and if it is holomorphic on $U \setminus \{a\}$ for some point $a \in U$, then f is holomorphic on U.*

Proof.

□

• *The zeta function $\zeta(s)$.* Using Morera's theorem we introduce the most important function of analytic number theory. For $a \in \mathbb{R}$ we define the half-planes

$$U_{>a} = \{z \in \mathbb{C}\colon\ \mathrm{re}(z) > a\} \ \text{ and } \ U_{\geq a} = \{z \in \mathbb{C}\colon\ \mathrm{re}(z) \geq a\}\,,$$

and similarly for the halfplanes $U_{<a}$ and $U_{\le a}$. Recall that for real $a > 0$ and $z \in \mathbb{C}$ we have $a^z := \exp(z \log a)$. For any $s \in U_{>1}$ we define *the zeta function* as the sum

$$\zeta(s) = \textstyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} n^{-s}\,.$$

The series absolutely converges because $|n^s| = n^{\mathrm{re}(s)}$.

Corollary 6 (defining $\zeta(s)$) *$\zeta(s)$ is holomorphic on $U_{>1}$.*

Proof. This follows from Corollary 4. Let $A \subset U_{>1}$ be compact. Then there is a $\delta > 0$ such that $A \subset U_{>1+\delta}$. Let an $\varepsilon > 0$ be given. Then there is n_0 such that for every $n \ge m \ge n_0$ we have $\sum_{j=m}^{n} j^{-1-\delta} \le \varepsilon$. Then for the same n and m and every $s \in A$,

$$\big|\textstyle\sum_{j=m}^{n} \frac{1}{j^s}\big| \le \sum_{j=m}^{n} \frac{1}{j^{\mathrm{re}(s)}} \le \sum_{j=m}^{n} \frac{1}{j^{1+\delta}} \le \varepsilon\,.$$

Thus the series defining $\zeta(s)$ converges uniformly on A. $\square$

• *Extending $\zeta(s)$.* The function $\zeta(s)$ has a meromorphic extension to $\mathbb{C} \setminus \{1\}$. For our purposes an extension to $U_{>0} \setminus \{1\}$ suffices.

Proposition 7 (extending $\zeta(s)$) *There exists a holomorphic function $f(s)\colon U_{>0} \to \mathbb{C}$ such that on $U_{>1}$ we have equality*

$$\zeta(s) = f(s) + (s-1)^{-1}\,.$$

The right-hand side extends $\zeta(s)$ to the meromorphic function

$$\zeta(s)\colon U_{>0} \setminus \{1\} \to \mathbb{C}\,.$$

Proof. We obtain a holomorphic function $f\colon U_{>0} \to \mathbb{C}$ such that $\zeta(s) - \frac{1}{s-1} = f(s)$ for every $s \in U_{>1}$. To this end we define, for $n \in \mathbb{N}$ and $s \in \mathbb{C}$ with $s \ne 1$, functions

$$g_n(s) = \textstyle\int_n^{n+1} (n^{-s} - x^{-s})\,\mathrm{d}x = \frac{1}{n^s} - \frac{1}{s-1}\big(\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}\big)\,.$$

The middle integral formula works also for $s = 1$ and shows that $g_n(s)\colon \mathbb{C} \to \mathbb{C}$ is continuous. The last algebraic formula shows that $g_n(s)$ is holomorphic on $\mathbb{C} \setminus \{1\}$. By Corollary 5 the function $g_n(s)$ is entire. The algebraic formula shows that for every $s \in U_{>1}$,

$$\zeta(s) - \tfrac{1}{s-1} = \textstyle\sum_{n=1}^{\infty} g_n(s)\,.$$

For $n \in \mathbb{N}$, $s \in \mathbb{C}$ and $x \in [n, n+1]$ an integral ML estimate gives the bound

$$|n^{-s} - x^{-s}| = \left|s \textstyle\int_n^x \frac{\mathrm{d}u}{u^{s+1}}\right| \le |s| \cdot 1 \cdot \frac{1}{n^{\mathrm{re}(s)+1}} = \frac{|s|}{n^{\mathrm{re}(s)+1}}\,.$$

Using an integral ML estimate again we get the bound

$$|g_n(s)| \le 1 \cdot \tfrac{|s|}{n^{\mathrm{re}(s)+1}} = \tfrac{|s|}{n^{\mathrm{re}(s)+1}}\,.$$

We may define $f(s) = \sum_{n=1}^{\infty} g_n(s)$ for any $s \in U_{>0}$ because by the bound on $|g_n(s)|$ this series absolutely converges. As in Corollary 6, this convergence is uniform on any compact set $A \subset U_{>0}$. By Corollary 4 the function $f(s)\colon U_{>0} \to \mathbb{C}$ is holomorphic and is therefore the desired function. $\square$

In the previous proof we made an effort to obtain the standard extension argument for $\zeta(s)$ in a completely clear and rigorous form.

• *The Euler product.* We denote by $p_1 = 2 < p_2 = 3 < \ldots$ the increasing sequence (p_n) of prime numbers.

Theorem 8 (Euler product for $\zeta(s)$) *For any $s \in U_{>1}$,*

$$\zeta(s) = \lim_{n\to\infty} \textstyle\prod_{j=1}^{n} \left(1 - p_j^{-s}\right)^{-1} =: \prod_p \frac{1}{1-1/p^s}\,.$$

Proof. We denote the above n-th partial product by $P(n, s)$. Let $n \in \mathbb{N}$ and $s \in U_{>1}$. Then

$$\begin{aligned} |\zeta(s) - P(n,\, s)| &= \left|\textstyle\sum_{m=1}^{\infty} \frac{1}{m^s} - \prod_{j=1}^{n} \sum_{m=0}^{\infty} (p_j^m)^{-s}\right| \\ &\le \textstyle\sum_{m \ge p_n} m^{-\mathrm{re}(s)} =: T(n,\, s)\,. \end{aligned}$$

We used the Fundamental Theorem of Arithmetic by which every natural number has a unique expression as a product

$$q_1^{a_1} q_2^{a_2} \ldots q_k^{a_k} \quad (a_i \in \mathbb{N})$$

of powers of distinct primes q_i. Since $\lim_{n\to\infty} T(n,s) = 0$ for every $s \in U_{>1}$, the Euler product for $\zeta(s)$ follows. $\Box$

• *The logarithmic derivative of* ζ. In this passage we rigorously deduce the formula that for any $s \in U_{>1}$,

$$\tfrac{\zeta'(s)}{\zeta(s)} = \textstyle\sum_p \tfrac{\log p}{1-p^s}\,.$$

It is usually obtained by taking logarithm of the Euler product and differentiating the result. It is a challenge to do this really rigorously because in the complex domain logarithm behaves badly. In fact, I did it in my LN cited on p. 1. Now, 15 years later, I take a different route.

Proposition 9 (ζ') *For any* $s \in U_{>1}$,

$$\zeta'(s) = \textstyle\sum_{n=1}^{\infty} \log n \cdot n^{-s}\,.$$

Proof.

$\Box$

Proposition 10 (product of Dirichlet series) *Let* $A(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ *and* $B(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ *be Dirichlet series, absolutely convergent on* $U_{>1}$*, and let* $c_n = \sum_{de=n} a_d b_e$*. Then* $C(s) = \sum_{n=1}^{\infty} c_n n^{-s}$ *absolutely converges on* $U_{>1}$ *and*

$$A(s) \cdot B(s) = C(s) \quad (s \in U_{>1})\,.$$

Proof.

□

Proposition 11 **(μ)** *For any $s \in U_{>1}$,*

$$\zeta(s) \cdot \sum_{n=1}^{\infty} \mu(n) \cdot n^{-s} = 1 \,.$$

Proof.

□

Corollary 12 **($\zeta \neq 0$ on $U_{>1}$)** *We have $\zeta(s) \neq 0$ for every s in $U_{>1}$ and*

$$\tfrac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \mu(n) \cdot n^{-s} \quad (s \in U_{>1}) \,.$$

Proof.

□

Proposition 13 **(ζ'/ζ)** *For any $s \in U_{>1}$,*

$$\tfrac{\zeta'(s)}{\zeta(s)} = -\sum_{n=1}^{\infty} \Lambda(n) \cdot n^{-s} = \sum_{p} \tfrac{\log p}{1-p^s} \,.$$

Proof.

□

• *Non-vanishing of $\zeta(s)$ on $U_{\geq 1}$.* In every analytic proof of PNT[1] the following property of $\zeta(s)$ is crucial.

Theorem 14 **($\zeta \neq 0$)** *For any $s \in U_{\geq 1} \setminus \{1\}$ we have $\zeta(s) \neq 0$.*

Proof. Will be added later. □

[1]This does not apply to the elementary proofs of PNT which do not use complex analysis.

Corollary 15 (extending ζ'/ζ) *The function*

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1}$$

has a holomorphic extension to some $U \supset U_{\geq 1}$.

Proof. Proposition 7 and Theorem **??** show that $\frac{\zeta'(s)}{\zeta(s)}$ extends holomorphically to some $U \supset (U_{\geq 1} \setminus 1)$. By Proposition 7, on $U_{>0} \setminus \{1\}$ we have expression $\zeta(s) = f(s) + \frac{1}{s-1}$ where $f(s)$ is holomorphic on $U_{>0}$. Then on a deleted open disc $B(1, \delta) \setminus \{1\}$ we have

$$\frac{\zeta'(s)}{\zeta(s)} + \frac{1}{s-1} = \frac{-(s-1)^{-2} + f'(s)}{(s-1)^{-1} + f(s)} + \frac{1}{s-1} = \frac{f(s) + (s-1)f'(s)}{1 + (s-1)f(s)} \,.$$

The latter fraction is holomorphic on $B(1, \delta)$. □

• *Newman's proof.* The contribution of D. J. Newman to PNT is in his simple proof of the following version of theorems obtained earlier by *Norbert Wiener (1894–1964)* and *Shikao Ikehara (1904–1984)*.

Theorem 16 (Wiener–Ikehara) *Let*

$$f\colon [0, +\infty) \to \mathbb{R}$$

be a bounded function that for every number $a > 0$ has the Riemann integral $\int_0^a f$. Let the holomorphic function

$$g(z) = \lim_{a \to +\infty} \int_0^a f(t) \exp(-zt) \, \mathrm{d}t \colon U_{>0} \to \mathbb{C}$$

have a holomorphic extension to some $U \supset U_{\geq 0}$. Then

$$\lim_{a \to +\infty} \int_0^a f = g(0) \,.$$

Before we plunge in the proof we justify that $g(z)\colon U_{>0} \to \mathbb{C}$ is correctly defined and is holomorphic. It follows from Morera's theorem (details will be added later). Now we can prove the theorem.

Proof. (Newman) For real $a > 0$ we set

$$g_a(z) = \int_0^a f(t) \exp(-zt) \,\mathrm{d}t \,.$$

By ... this is an entire function. We show that

$$\lim_{a \to +\infty} g_a(0) = g(0) \,.$$

For real $R, \delta > 0$ we consider the set

$$C(R,\, \delta) = \{z \in \mathbb{C}\colon\ |z| \leq R \wedge \mathrm{re}(z) \geq -\delta\} \quad (\subset \mathbb{C}) \,,$$

where $\delta = \delta(R)$ is so small that $g(z)$ extends holomorphically to an open set containing $C(R, \delta)$; for every $R > 0$ such $\delta > 0$ exists due to the assumption on $g(z)$ and compactness of the half-disc

$$\{z \in \mathbb{C}\colon\ |z| \leq R \wedge \mathrm{re}(z) \geq 0\} \,.$$

Let $C = C(R)$ be the boundary $\partial C(R, \delta)$. By the Cauchy formula,

$$\begin{aligned} g(0) - g_a(0) &= \tfrac{1}{2\pi i} \int_C (g(z) - g_a(z)) \exp(za) \big(1 + z^2 R^{-2}\big) z^{-1} \,\mathrm{d}z \\ &=: \tfrac{1}{2\pi i} \int_C (g(z) - g_a(z)) G(z) = \tfrac{1}{2\pi i} I(R,\, a) \,. \end{aligned}$$

In order to show that $I(R, a) \to 0$ as $a \to +\infty$, we express the integral $I(R, a)$ as a sum of three contributions which we separately estimate. With $C^- = C \cap U_{\leq 0}$, $K = \{z \in \mathbb{C}\colon\ |z| = R,\ \mathrm{re}(z) \leq 0\}$ and $C^+ = C \cap U_{\geq 0}$ we define

$$\begin{aligned} I(R,\, a) &= I_1(R,\, a) + I_2(R,\, a) + I_3(R,\, a) \\ &:= \int_{C^+} (g(z) - g_a(z)) G(z) + \int_{C^-} g(z) G(z) - \\ &\quad - \int_K g_a(z) G(z) \,. \end{aligned}$$

In $I_3(R, a)$ we could replace C^- with the half-circle K without changing the integral because the integrand is holomorphic on $\mathbb{C} \setminus \{0\}$.

The integral $I_1(R,a) = \int_{C^+}(g(z) - g_a(z))G(z)$. Let $B \geq 0$ be such that $|f(t)| \leq B$ for every $t \geq 0$. For $z \in U_{\geq 0}$ we have

$$|g(z) - g_a(z)| \leq B \int_a^{+\infty} \left|\mathrm{e}^{-tz}\right| \mathrm{d}t = \tfrac{B\mathrm{e}^{-\mathrm{re}(z)\cdot a}}{\mathrm{re}(z)}\,.$$

For $z \in \mathbb{C}$ with $|z| = R$ we have

$$|G(z)| = \left|\tfrac{\mathrm{e}^{za}(z+\overline{z})}{R^2}\right| = 2\mathrm{e}^{\mathrm{re}(z)a} \cdot |\mathrm{re}(z)| \cdot R^{-2}\,.$$

The curve C^+ has length πR and we get the ML estimate

$$|I_1(R,\, a)| \leq \tfrac{2\pi B}{R}\,.$$

The integral $I_3(R,a) = \int_K g_a(z)G(z)$. For $z \in U_{\leq 0}$ we have

$$|g_a(z)| \leq \left|\,\int_0^a f(t)\mathrm{e}^{-tz}\,\mathrm{d}t\right| \leq B\int_{-\infty}^a \left|\mathrm{e}^{-tz}\,\mathrm{d}t\right| = \tfrac{B\mathrm{e}^{-\mathrm{re}(z)\cdot a}}{|\mathrm{re}(z)|}\,.$$

The curve K has length πR and we get the same ML estimate

$$|I_3(R,\, a)| \leq \tfrac{2\pi B}{R}\,.$$

The integral $I_2(R,a) = \int_{C^-} g(z)G(z)$. We write

$$I_2(R,\, a) = \int_{C^-} g(z)z^{-1}(1 + z^2R^{-2}) \cdot \mathrm{e}^{za} =: \int_{C^-} J(z) \cdot \mathrm{e}^{za}\,.$$

Let $M_1 = M_1(R) = \max_{C^-} |J(z)|$. Then

$$|I_2(R,\, a)| \leq M_1 \int_{C^-} \left|\mathrm{e}^{za}\right| \mathrm{d}z\,.$$

From the definition of C^- we see that for every $\varepsilon > 0$ there is a $\kappa > 0$ such that on C^- we have $\left|\mathrm{e}^{za}\right| \leq \mathrm{e}^{-\kappa a}$, except the part of C^- near to the imaginary axis whose length is the ε-fraction of the length $|C^-| \leq 3R$. On this part of C^- we use the trivial bound $\left|\mathrm{e}^{za}\right| \leq 1$. Thus

$$|I_2(R,\, a)| \leq M_1\big(\mathrm{e}^{-\kappa a} + \varepsilon\big) \cdot |C^-| \leq 3M_1R\big(\mathrm{e}^{-\kappa a} + \varepsilon\big)\,.$$

Hence for every fixed $R > 0$ we have $\lim_{a\to+\infty} |I_2(R,a)| = 0$.

We combine these three bounds. Let an $\varepsilon > 0$ be given. We fix an $R > 8\pi\frac{B}{\varepsilon}$ and the corresponding curve $C = C(R)$. Then $|I_1(R,a)| + |I_3(R,a)| \le \frac{\varepsilon}{2}$ for every a. Then we take an $a_0 \ge 0$ such that if $a \ge a_0$ then $|I_2(R,a)| \le \frac{\varepsilon}{2}$. For any such a we have

$$|I(R,\,a)| \le |I_1(R,\,a)| + |I_3(R,\,a)| + |I_2(R,\,a)| \le \tfrac{\varepsilon}{2} + \tfrac{\varepsilon}{2} = \varepsilon\,.$$

□

• *Extending* $\frac{F(z+1}{z+1} - \frac{1}{z}$. We introduce the function

$$F(s) = \textstyle\sum_p \frac{\log p}{p^s}\colon U_{>1} \to \mathbb{C}\,.$$

By Corollary 4 the function $F(s)$ is holomorphic.

Proposition 17 (an extension) *The holomorphic function*

$$\tfrac{F(z+1)}{z+1} - \tfrac{1}{z}\colon U_{>0} \to \mathbb{C}$$

has a holomorphic extension to some $U \supset U_{\ge 0}$.

Proof. For $s \in U_{>1}$ we have by Corollary **??** that

$$-\tfrac{\zeta'(s)}{\zeta(s)} = \textstyle\sum_p \frac{\log p}{p^s-1} = F(s) + \sum_p \frac{\log p}{p^s(p^s-1)}\,.$$

Thus on $U_{>1}$,

$$F(s) = -\tfrac{\zeta'(s)}{\zeta(s)} - \textstyle\sum_p \frac{\log p}{p^s(p^s-1)}\,.$$

By Corollary 4, the sum is holomorphic on $U_{>1/2}$. By Corollary 15, the function $F(s) - (s-1)^{-1}$ has holomorphic extension to some $U \supset U_{\ge 1}$. □

• *Convergence of the integral* $\int_1^{+\infty}(\vartheta(x) - x)x^{-2}\,\mathrm{d}x$. We deduce from the previous theorem the existence and finiteness of the next limit.

Proposition 18 (convergence of an $\int$) *The limit*

$$\alpha := \lim_{a\to+\infty} \int_1^a (\vartheta(x)-x)x^{-2} \quad (\in \mathbb{R})$$

exists and is finite.

Proof. For any $s \in U_{>1}$,

$$\begin{aligned} s\int_0^{+\infty} \vartheta(\mathrm{e}^t)\mathrm{e}^{-st}\,\mathrm{d}t &= s\int_1^{+\infty} \vartheta(x)x^{-s-1}\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} \vartheta(n)\cdot s\int_n^{n+1} x^{-s-1}\,\mathrm{d}x \\ &= \sum_{n=1}^{\infty} \vartheta(n)\big(n^{-s}-(n+1)^{-s}\big) \\ &= \sum_{n=1}^{\infty} n^{-s}(\vartheta(n)-\vartheta(n-1)) \\ &= \sum_p \tfrac{\log p}{p^s} = F(s)\,. \end{aligned}$$

We set $s = z+1$, divide by $z+1$, subtract $\frac{1}{z} = \int_0^{+\infty} \mathrm{e}^{-zt}\,\mathrm{d}t$ and get that

$$\int_0^{+\infty} \big(\vartheta(\mathrm{e}^t)\mathrm{e}^{-t}-1\big)\mathrm{e}^{-zt}\,\mathrm{d}t = \tfrac{F(z+1)}{z+1} - \tfrac{1}{z}\,.$$

By Propositions 2 and 17, the functions $f(t) = \vartheta(\mathrm{e}^t)\mathrm{e}^{-t}-1$ and $g(z) = F(z+1)(z+1)^{-1} - z^{-1}$ satisfy assumptions of Theorem 16, which gives

$$\lim_{a\to+\infty} \int_0^{\log a} f(t) = \lim_{a\to+\infty} \int_1^a (\vartheta(x)-x)x^{-2} = g(0) =: \alpha\,.$$

□

Corollary 19 (a Cauchy condition) *For every $\varepsilon > 0$ there is a $c \ge 1$ such that for every $a, b \in \mathbb{R}$ with $b \ge a \ge c$ we have*

$$\Big|\int_a^b (\vartheta(x)-x)x^{-2}\Big| \le \varepsilon\,.$$

Proof. Let $f(x) = (\vartheta(x)-x)x^{-2}$ and let an $\varepsilon > 0$ be given. By Proposition 18 there is a $c \ge 1$ such that if $a \ge c$ then $\big|\int_1^a f - \alpha\big| \le$

$\frac{\varepsilon}{2}$. We have by the additivity of integrals and the triangle inequality that for every $a, b \in \mathbb{R}$ with $b \geq a \geq c$,

$$\left| \int_a^b f \right| = \left| \int_1^b f - \int_1^a f \right| \leq \left| \int_1^b f - \alpha \right| + \left| \alpha - \int_1^a f \right| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \, .$$

□

- *Conclusion: $\vartheta(x) \sim x$ ($x \to +\infty$).* We finish the proof of PNT.

Proposition 20 ($\vartheta(x) \sim x$) $\lim_{x \to +\infty} \vartheta(x) x^{-1} = 1$.

Proof. Suppose, for the contrary, that there is a $\lambda > 1$ such that $\frac{\vartheta(x)}{x} \geq \lambda$ for arbitrarily large $x > 0$. Then we have for any such x, since $\vartheta(x)$ weakly increases, that

$$\int_x^{\lambda x} (\vartheta(t) - t) t^{-2} \geq \int_x^{\lambda x} (\lambda x - t) t^{-2} = \int_1^{\lambda} \frac{\lambda - u}{u^2} =: d > 0 \quad (u = \tfrac{t}{x}) \, .$$

This contradicts Corollary 19. If there is a $\lambda \in (0, 1)$ such that $\frac{\vartheta(x)}{x} \leq \lambda$ for arbitrarily large $x > 0$, we get a similar contradiction $\ldots$ $d < 0$ by bounding the integral over the interval $[\lambda x, x]$. □

In view of the initial Proposition 1, this concludes the proof of PNT.

THANK YOU FOR YOUR ATTENTION!

No homework exercises in this lecture.