

MATHEMATICAL ANALYSIS 3 (NMAI056)

summer term 2025/26

lecturer: Martin Klazar

**LECTURE 10 (April 27, 2026) INTRODUCTION TO
COMPLEX ANALYSIS 2**

- *The Cauchy–Goursat theorem for rectangles.* This is the main result of this lecture: (R is a rectangle)

$\int_{\partial R} f = 0$ whenever $R \subset U$ and $f: U \rightarrow \mathbb{C}$ is holomorphic.

- *Linear functions.* We first prove it in the case that $f(z)$ is a linear function. Let $k \in \mathbb{N}$ and $u = ab$ ($\subset \mathbb{C}$) be a segment. The *k-equipartition* of u is the partition of u into k subsegments of the same length $\frac{1}{k}|u|$. It is the image of the partition $0 < \frac{1}{k} < \frac{2}{k} < \dots < \frac{k-1}{k} < 1$ of the interval $I = [0, 1]$.

Exercise 1 Let $a, b, \alpha, \beta \in \mathbb{C}$ with $a \neq b$. Prove from the definition of the Cauchy integral that

$$\int_{ab}(\alpha z + \beta) = \alpha\left(\frac{b^2}{2} - \frac{a^2}{2}\right) + \beta(b - a) = g(b) - g(a),$$

where $g(z) = \frac{1}{2}\alpha z^2 + \beta z$. Hint – use equipartitions of ab .

Corollary 2 (the easy C.–G. theorem) Let $\alpha, \beta \in \mathbb{C}$ and let $R \subset \mathbb{C}$ be a rectangle. Then

$$\int_{\partial R}(\alpha z + \beta) = 0.$$

Proof. Let a, b, c, d be the canonical vertices of R and $f(z) = \alpha z + \beta$. By the definition of $\int_{\partial R}$ and the previous exercise, $\int_{\partial R} f$ is $g(b) - g(a) + g(c) - g(b) + g(d) - g(c) + g(a) - g(d) = 0$. \square

- *Contour integration.* We express the Cauchy integral $\int_u f$ in terms of real Riemann integrals.

Exercise 3 *Prove the next proposition.*

Proposition 4 *Let $a, b \in \mathbb{C}$, $a \neq b$, $f: ab \rightarrow \mathbb{C}$ be continuous and let $\varphi(t) = t(b - a) + a: [0, 1] \rightarrow \mathbb{C}$ be the parametrization of the segment $u = ab$. Then*

$$\begin{aligned} \int_u f &= \int_0^1 f(\varphi(t)) \cdot \varphi'(t) dt = (b - a) \int_0^1 f(\varphi(t)) dt \\ &= (b - a) \left(\int_0^1 \operatorname{re}(f(\varphi(t))) dt + i \cdot \int_0^1 \operatorname{im}(f(\varphi(t))) dt \right) \end{aligned}$$

– *except for the first \int , all other are Riemann \int s.*

To relate our approach to the standard one, we define the contour integral $\int_\varphi f$ for any curve φ . Let $f: U \rightarrow \mathbb{C}$ and $\varphi: [a, b] \rightarrow U$ be a continuous and piece-wise smooth function. The *integral of the function f along the curve φ* is given by

$$\begin{aligned} \int_\varphi f &= \int_a^b f(\varphi(t)) \cdot \varphi'(t) dt \\ &= \int_a^b \operatorname{re}(f(\varphi(t)) \cdot \varphi'(t)) dt + i \cdot \int_a^b \operatorname{im}(f(\varphi(t)) \cdot \varphi'(t)) dt, \end{aligned}$$

if the last two real Riemann integrals exist. By Proposition 4, the Cauchy integral \int_u is a particular case of \int_φ .

Exercise 5 *Let $\varphi(t) = e^{2\pi it}: [0, \frac{1}{2}] \rightarrow \mathbb{C}$ be given by*

$$\varphi(t) = \sum_{n=0}^{\infty} \frac{(2\pi it)^n}{n!}.$$

It is a parametrization of the upper unit semicircle. Let $f(z) = z^2$. Find $\int_\varphi f$.

- *The constant ρ ($= 2\pi i$).* If the constant ρ defined in the next theorem were 0, the complex analysis would collapse.

Theorem 6 *Let S be the square with the vertices $\pm 1 \pm i$. Then $\rho = \int_{\partial S} \frac{1}{z} \neq 0$, in fact $\text{im}(\rho) \geq 4$.*

Proof. The canonical vertices of S are $a = -1 - i$, $b = 1 - i$, $c = 1 + i$ and $d = -1 + i$. Let

$$p_n = (a_0, a_1, \dots, a_n)$$

be the n -equipartition of ab . Multiplication by i rotates p_n counterclockwise around the origin by $\frac{1}{2}\pi$:

$$q_n = ip_n = (ia_0, ia_1, \dots, ia_n)$$

is the n -equipartition of bc . Similarly, $r_n = iq_n = -p_n$ is the n -equipartition of cd and $s_n = ir_n = -ip_n$ is the n -equipartition of da . But for $f(z) = \frac{1}{z}$ we have

$$C(f, p_n) = C(f, q_n) = C(f, r_n) = C(f, s_n).$$

Indeed, extending the fraction by i gives

$$\begin{aligned} C(f, p_n) &= \sum_{j=1}^n \frac{(b-a)/n}{a+j(b-a)/n} = \sum_{j=1}^n \frac{(ib-ia)/n}{ia+j(ib-ia)/n} \\ &= \sum_{j=1}^n \frac{(c-b)/n}{b+j(c-b)/n} = C(f, q_n), \end{aligned}$$

and similarly for the other two equalities. As $b - a = 2$ and $a = -1 - i$, extending the fraction by $\frac{2j}{n} - 1 + i$ we get

$$\begin{aligned} \text{im}(C(f, p_n)) &= \text{im}\left(\sum_{j=1}^n \frac{2/n}{-1-i+2j/n}\right) = \text{im}\left(\frac{2}{n} \sum_{j=1}^n \frac{2j/n-1+i}{(2j/n-1)^2+1}\right) \\ &= \frac{2}{n} \sum_{j=1}^n \frac{1}{(2j/n-1)^2+1} \geq \frac{2}{n} \sum_{j=1}^n \frac{1}{2} = 1. \end{aligned}$$

By Exercise 7,

$$\text{im}\left(\int_{\partial S} \frac{1}{z}\right) = 4 \cdot \text{im}\left(\int_{ab} \frac{1}{z}\right) = 4 \cdot \lim_{n \rightarrow \infty} \text{im}(C(\frac{1}{z}, p_n)) \geq 4 \cdot 1 = 4.$$

Indeed $\rho \neq 0$. □

Exercise 7 Let $(z_n) \subset \mathbb{C}$ be a convergent sequence. Prove that $\operatorname{im}(\lim_{n \rightarrow \infty} z_n) = \lim_{n \rightarrow \infty} \operatorname{im}(z_n)$.

Exercise 8 The previous proof gives that $\operatorname{re}(\rho) = 0$.

Exercise 9 ($\rho = 2\pi i$) Again, let $a = -1 - i$ and $b = 1 - i$. Compute by Proposition 4 that $\int_{ab} \frac{1}{z} = \frac{1}{2}\pi i$. Thus, by the previous proof,

$$\rho = 4 \cdot \frac{1}{2}\pi i = 2\pi i.$$

Exercise 10 Let $\varphi(t): [0, 1] \rightarrow \mathbb{C}$, $\varphi(t) = e^{2\pi i t}$ and $f(z) = \frac{1}{z}$. Show that $\int_{\varphi} f = 2\pi i$.

• *Nested sets and quarters.* Recall that a set $X \subset \mathbb{C}$ has *diameter* $\operatorname{diam}(X) = \sup(\{|x - y|: x, y \in X\}) \in [0, +\infty) \cup \{+\infty\}$.

Exercise 11 Let $A_n \neq \emptyset$ be closed sets such that $\mathbb{C} \supset A_1 \supset A_2 \supset \dots$ and $\lim \operatorname{diam}(A_n) = 0$. Then

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset.$$

Hint – see the proof of Baire's theorem.

Let R be a rectangle with the canonical vertices (a, b, c, d) . We define the *quarters* of R . Let $e = \frac{a+b}{2}$, $f = \frac{b+c}{2}$, $g = \frac{c+d}{2}$ and $h = \frac{d+a}{2}$ be the midpoints of the sides of R and $j = \frac{a+c}{2}$ be the center point of R . The four *quarters* of R are the rectangles A , B , C and D with the canonical vertices, respectively,

$$(a, e, j, h), (e, b, f, j), (j, f, c, g) \text{ and } (h, j, g, d).$$

So R is divided into four quarters by cutting along the segments eg and hf . For each quarter E we have

$$\operatorname{per}(E) = \frac{1}{2} \cdot \operatorname{per}(R) \text{ and } \operatorname{diam}(E) = \frac{1}{2} \cdot \operatorname{diam}(R).$$

- *The Cauchy–Goursat theorem for rectangles.* This is the next theorem, which we already stated at the beginning.

Theorem 12 (the Cauchy–Goursat thm for rectangles)

Let $f: U \rightarrow \mathbb{C}$ be holomorphic and let $R \subset U$ be a rectangle. Then

$$\int_{\partial R} f = 0.$$

Proof. Let f , U and R be as stated. We define rectangles

$$R = R_0 \supset R_1 \supset R_2 \supset \dots$$

such that always R_{n+1} is a quarter of R_n and

$$\left| \int_{\partial R_{n+1}} f \right| \geq \frac{1}{4} \left| \int_{\partial R_n} f \right|. \quad (1)$$

Suppose that R_0, R_1, \dots, R_n have been already defined. Let A, B, C and D be the quarters of the rectangle R_n . We claim that

$$\int_{\partial R_n} f = \int_{\partial A} f + \int_{\partial B} f + \int_{\partial C} f + \int_{\partial D} f. \quad (2)$$

This follows from parts 3 and 4 of the last theorem in the previous lecture. We express each of the integrals $\int_{\partial A} f$, $\int_{\partial B} f$, $\int_{\partial C} f$, and $\int_{\partial D} f$ as the sum of four integrals over the sides. On the right-hand side of the equality (2), we get 16 terms. Eight of them correspond to the sides of quarters lying inside R_n . They mutually cancel out because they form four pairs of opposite orientations of the same segment. The remaining eight terms correspond to the sides of the quarters lying on ∂R_n . These add to the integral on the left-hand side of the equality (2). Inequalities (1) follow from the triangle inequality: for some quarter $E \in \{A, B, C, D\}$, we have

$$\left| \int_{\partial E} f \right| \geq \frac{1}{4} \cdot \left| \int_{\partial R_n} f \right|$$

and can set $R_{n+1} = E$.

By Exercise 11 there exists a point z_0 such that

$$z_0 \in \bigcap_{n=0}^{\infty} R_n.$$

Since $R_0 = R \subset U$, we have $z_0 \in U$. Let an $\varepsilon > 0$ be given. Since $f'(z_0)$ exists, there is a $\delta > 0$ and a function

$$\Delta: B(z_0, \delta) \rightarrow \mathbb{C}$$

such that $B(z_0, \delta) \subset U$, that on $B(z_0, \delta)$ we have $|\Delta(z)| \leq \varepsilon$, and that

$$f(z) = \underbrace{f(z_0) + f'(z_0) \cdot (z - z_0)}_{g(z)} + \underbrace{\Delta(z) \cdot (z - z_0)}_{h(z)}.$$

We consider functions $g(z)$ and $h(z)$. It is clear that $g(z)$ is linear and $h(z)$ ($= f(z) - g(z)$) is continuous on $B(z_0, \delta)$. Let $n \in \mathbb{N}_0$ be so large that $R_n \subset B(z_0, \delta)$ – only here we need that $\lim \text{diam}(R_n) = 0$ (for the existence of the point z_0 it is not essential, see Exercise 14). By the linearity of the integral $\int_{\partial R}$ and Corollary 2 we have that

$$\int_{\partial R_n} f = \int_{\partial R_n} g + \int_{\partial R_n} h \stackrel{\text{Cor. 2}}{=} \int_{\partial R_n} h. \quad (3)$$

Thus

$$\begin{aligned} \left| \int_{\partial R_n} h \right| &\stackrel{\text{ML estimate}}{\leq} \max_{z \in \partial R_n} |\Delta(z) \cdot (z - z_0)| \cdot \text{per}(R_n) \\ &\leq \varepsilon \cdot \text{diam}(R_n) \cdot \text{per}(R_n) \\ &= \varepsilon \cdot \frac{\text{diam}(R)}{2^n} \cdot \frac{\text{per}(R)}{2^n} \\ &\leq \varepsilon \cdot \frac{\text{per}(R)^2}{4^n}. \end{aligned} \quad (4)$$

We used that quartering halves diameters and perimeters, and that the diameter of a rectangle is smaller than its perimeter. According

to the previous results we have that

$$\frac{1}{4^n} \left| \int_{\partial R} f \right| \stackrel{\text{ineq. (1)}}{\leq} \left| \int_{\partial R_n} f \right| \stackrel{\text{eq. (3)}}{=} \left| \int_{\partial R_n} h \right| \stackrel{\text{ineq. (4)}}{\leq} \varepsilon \cdot \frac{\text{per}(R)^2}{4^n}.$$

Hence $\left| \int_{\partial R} f \right| \leq \varepsilon \cdot \text{per}(R)^2$. It holds for every $\varepsilon > 0$, and we have $\int_{\partial R} f = 0$. \square

Exercise 13 *What is the value of the function $\Delta(z)$ in the proof at $z = z_0$?*

Exercise 14 *Prove that for non-emptiness of the intersection in Exercise 11, it suffices to assume that the set A_1 is bounded (instead of the zero limit of diameters). Show that it does not hold in general metric spaces.*

The theorem originates with the French mathematician *Augustin-Louis Cauchy (1789–1857)*. For several years he lived in Prague in political exile. Cauchy assumed continuity of the derivative f' . It was *Édouard Goursat (1858–1936)* who proved the theorem in 1900 only with the assumption of existence f' , in the article E. Goursat, Sur la définition générale des fonctions analytiques, d'après Cauchy, *Trans. Amer. Math. Soc.* **1** (1900), 14–16.

The C.–G theorem for rectangles suffices for our purposes. The theorem holds for general curves and we only outline the proof.

Theorem 15 (Cauchy–Goursat) *Let $f: U \rightarrow \mathbb{C}$ be a holomorphic function and $\varphi: [a, b] \rightarrow U$ be a continuous and piecewise smooth function. Suppose that φ is injective, with the exception of $\varphi(a) = \varphi(b)$, and that the interior of φ — the bounded connected component of $\mathbb{C} \setminus \varphi[[a, b]]$ — is a subset of U . Then*

$$\int_{\varphi} f = 0.$$

Sketch of the proof. We draw in \mathbb{C} , with the help of horizontal and vertical lines, a fine square grid \mathcal{M} . A simple closed curve $\psi: [a, b] \rightarrow U$ runs along the sides of the grid \mathcal{M} and satisfies that (i) for a given $\varepsilon > 0$ it holds that $|\int_{\varphi} f - \int_{\psi} f| \leq \varepsilon$ (the curve ψ closely approximates the curve φ) and (ii) the interior of the curve ψ is a subset of the set U . Then

$$\int_{\psi} f = \sum_{R \in M} \int_{\partial R} f = \sum_{R \in M} 0 = 0,$$

where M is the set of elementary rectangles of the grid \mathcal{M} lying inside the curve ψ . The first equality holds for the same reason as equality (2) and the first equality in (5) below. The second equality follows from (ii) and the preceding C.-G. theorems for rectangles. By (i), $|\int_{\varphi} f| \leq \varepsilon$. This is true for every ε and $\int_{\varphi} f = 0$. \square

Exercise 16 *As in Exercise 10, let $\varphi(t): [0, 1] \rightarrow \mathbb{C}$, $\varphi(t) = e^{2\pi it}$ (now we parameterize the whole unit circle) and $f(z) = z^k$, where $k \in \mathbb{Z} \setminus \{-1\}$. Show that*

$$\int_{\varphi} f = 0.$$

This does not follow from the C.-G. theorem!

• *Independence of $\int_{\partial R} f$ on R .* We show that in some situations, the integral $\int_{\partial R} f$ does not depend much on the rectangle R . Recall that compact sets $A \subset \mathbb{C}$ are exactly the closed and bounded sets.

Proposition 17 (independence of $\int_{\partial R} f$ on R) *Let $R, S \subset \mathbb{C}$ be rectangles, let $A \subset \text{int}(R) \cap \text{int}(S)$ be a compact set, and let $f: \mathbb{C} \setminus A \rightarrow \mathbb{C}$ be a holomorphic function. Then*

$$\int_{\partial R} f = \int_{\partial S} f.$$

Proof. Let A , R , S , and f be as given, and let first $S \subset \text{int}(R)$. By extending the sides of S , we divide R into nine rectangles R_1, R_2, \dots, R_8, S . Then indeed

$$\int_{\partial R} f \stackrel{\text{as in (2)}}{=} \sum_{j=1}^8 \int_{\partial R_j} f + \int_{\partial S} f \stackrel{\text{Thm. 12, } R_j \subset \mathbb{C} \setminus A}{=} \int_{\partial S} f. \quad (5)$$

We reduce the case of general rectangles R and S to the previous case. By Exercises 18 and 19, for any rectangles R and S and any nonempty compact set $A \subset \text{int}(R) \cap \text{int}(S)$ there is a rectangle T such that

$$A \subset \text{int}(T) \quad \text{and} \quad T \subset \text{int}(R) \cap \text{int}(S).$$

By the already proven case,

$$\int_{\partial R} f = \int_{\partial T} f = \int_{\partial S} f.$$

□

Exercise 18 *Every nonempty intersection of two rectangles is a rectangle.*

Exercise 19 *For every two rectangles R and S and every nonempty compact set $A \subset \text{int}(R) \cap \text{int}(S)$ there is a rectangle T such that*

$$A \subset \text{int}(T) \quad \text{and} \quad T \subset \text{int}(R) \cap \text{int}(S).$$

THANK YOU FOR YOUR ATTENTION!

Homework Exercises. Please send to me (klazar@kam.mff.cuni.cz) by the end of the coming Sunday solutions to the Exercises 1, 5, 9, 16 and 19.