Mathematical Analysis 2 (MA2) – Lecture 6

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Complete texts of the lectures are on `https://kam.mff.cuni.cz/~klazar/MAII26en.html` and students are supposed to study them carefully; these slides are just outlines.

This lecture

1. Limits and order
2. Squeeze theorem in metric spaces
3. Properties of continuous maps
4. Multivariate elementary functions
5. More properties of continuous maps

Limits vs order 1

For sets $A, B \subset \mathbb{R}$, we write $A < B$ if $a < b$ for every $a \in A$ and every $b \in B$. Recall the balls in $\mathbb{R}$: $B_{\mathbb{R}}(a, \varepsilon) = (a - \varepsilon, a + \varepsilon)$, $B_{\mathbb{R}}(+\infty, \varepsilon) = (\varepsilon^{-1}, +\infty)$, and $B_{\mathbb{R}}(-\infty, \varepsilon) = (-\infty, -\varepsilon^{-1})$. Recall that $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$.

Theorem (limits vs order)

Let $M = \langle M, d \rangle$ and $N = \langle N, e \rangle$ be metric spaces, $X \subset M$, $Y \subset N$, $f : X \to \mathbb{R}$, $g : Y \to \mathbb{R}$. Let $b \in M$ be a limit point of X and $c \in N$ a limit point of Y. Assume $\lim_{x \to b} f(x) = C$ and $\lim_{y \to c} g(y) = D$ where $C, D \in \mathbb{R}^*$.

1. *If $C < D$, then $\exists\, \delta > 0$ such that $f[C_M(b, \delta)] < g[C_N(c, \delta)]$.*
2. *If $\forall \delta > 0\, \exists x \in C_M(b, \delta) \cap X\, \exists y \in C_N(c, \delta) \cap Y$ with $f(x) \geq g(y)$, then $C \geq D$.*

Proof.

(Sketch) 1. Easy: $\exists \varepsilon > 0$ such that $B_{\mathbb{R}}(C, \varepsilon) < B_{\mathbb{R}}(D, \varepsilon)$. 2. Part 1 has the form $\varphi \Rightarrow \psi$, the formula $\varphi \Rightarrow \psi \iff \neg\psi \Rightarrow \neg\varphi$ is a tautology, and part 2 is exactly $\neg\psi \Rightarrow \neg\varphi$. □

Limits vs order 2

The squeeze theorem is a popular result on limits of both real sequence and real functions. We can generalize it to maps between metric spaces as follows.

Theorem (generalized squeeze theorem)

Let $M = \langle M, d\rangle$ and $N = \langle N, e\rangle$ be metric spaces, $X \subset M$, $f, g\colon X \to N$, and let $b \in M$ be a limit point of X. If $\lim_{x\to b} f(x) = c \in N$ and if $\underline{\lim_{x\to b} e(f(x), g(x)) = 0}$, then also

$$\lim_{x\to b} g(x) = c\,.$$

Proof.

(Sketch) Simple application of the definition of limit. □

Exercise: in which metric spaces does the underlined limit take place?

Limits vs order 3

The classical squeeze theorem has the following counterpart for maps from a metric space to the real numbers.

Corollary (classical squeeze theorem)

Let $M = \langle M, d \rangle$ be a metric space, $X \subset M$, and $f, g, h \colon X \to \mathbb{R}$. Let b be a limit point of X and assume $\lim_{x \to b} f(x) = \lim_{x \to b} h(x) = c \in \mathbb{R}$. If

$$\exists \delta > 0 \ \forall a \in C_M(b, \delta) \cap X \colon \ f(a) \leq g(a) \leq h(a),$$

then $\lim_{x \to b} g(x) = c$.

Proof.

(Sketch) We use the previous theorem with $N = \mathbb{R}$, and the lemma below. □

Exercise: prove the lemma that if $a \leq b$ and c are real numbers such that $c \in [a, b]$, then $|c - a|, |c - b| \leq b - a$.

The arithmetic of continuity 1

Recall that $f\colon M \to N$ is continuous at $b \in M$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that $f[B_M(b,\delta)] \subset B_N(f(b),\varepsilon)$. Recall the Heine-style equivalent form of pointwise continuity.

Proposition (Heine's continuity)

Let $M = \langle M, d\rangle$ and $N = \langle N, e\rangle$ be metric spaces and $f\colon M \to N$. Then f is continuous at a point $b \in M$ iff for every sequence $(a_n) \subset M$ with $\lim a_n = b$, also $\lim f(a_n) = f(b)$.

As an application, we get the following.

Proposition (arithmetic of pointwise continuity in $\mathbb{R}$)

Let $M = \langle M, d\rangle$ be a metric space, $b \in M$, and $f, g\colon M \to \mathbb{R}$. If f and g are continuous at b, then $f + g$ and fg are continuous at b. If moreover $g(b) \neq 0$, then f/g is continuous at b.

Proof.

(Sketch) We use the previous prop. and the earlier T. 15 in l. 5 (AL of real sequences). □

The arithmetic of continuity 2

Proposition (arithmetic of pointwise continuity in $\mathbb{R}^m$)

Let $m \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, let $M = \langle M, d \rangle$ be a metric space, $b \in M$ be a point, and let $f, g \colon M \to \mathbb{R}^m$ be maps. If f and g are continuous at b, then so is the map $\alpha f + \beta g$.

Proof.

Exercise. □

Proposition (composition preserves pointwise continuity)

Let M, N, P be metric spaces, $g \colon M \to N$, $f \colon N \to P$, and let $b \in M$. If g is continuous at b and f at $g(b)$, then $f(g)$ is continuous at b.

Proof.

(Sketch) This is a nice application of Heine's continuity. □

Multivariate elementary functions 1

Some mathematical analysis courses discuss so called *elementary functions*. This a family of functions $f\colon X \to \mathbb{R}$ with $X \subset \mathbb{R}$ that can be obtained by repeated addition, multiplication, division, and composition, starting from some *basic functions*. We generalize this construction to multivariate real functions.

Definition (basic functions)

The following univariate real functions are basic.

1. $\exp x\colon \mathbb{R} \to (0, +\infty)$ and $\log x\colon (0, +\infty) \to \mathbb{R}$.
2. $\sin x\colon \mathbb{R} \to [-1, 1]$ and $\arcsin x\colon [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$.
3. Constant functions: $k_c(x)\colon \mathbb{R} \to \{c\}$ for any $c \in \mathbb{R}$.
4. Real power functions: $x^b\colon [0, +\infty) \to [0, +\infty)$ for any real $b > 0$.

Real power functions cannot be expressed by exp and log because, for example, $\sqrt{x} = x^{1/2}$ has the definition domain $[0, +\infty)$, but $\exp(\frac{1}{2} \log x)$ $(= \sqrt{x}$ for $x > 0)$ has the definition domain $(0, +\infty)$. Thus we set $0^b := 0$ for every $b > 0$.

Multivariate elementary functions 2

However, all these functions are univariate, and to get multivariate elementary functions, we need some multivariate basic functions. These are projections.

Definition (projections)

Let $n \in \mathbb{N}$ and $i \in \{1, \dots, n\}$,. We define the *i-th projection* $p_{n,i}(x_1, \dots, x_n) \colon \mathbb{R}^n \to \mathbb{R}$ by $p_{n,i}(x_1, \dots, x_n) := x_i$.

Exercise: projections are continuous. The arithmetic with functions works as follows.

Definition (sum, product, ratio)

For $f \colon A \to \mathbb{R}$ and $g \colon B \to \mathbb{R}$ with $A, B \subset \mathbb{R}^n$, we define on $A \cap B$ the functions

$$(f+g)(a) = f(a) + g(a) \text{ and } (fg)(a) = f(a)g(a)\,.$$

We define $(f/g)(a) = f(a)/g(a)$ on $(A \cap B) \setminus Z(g)$ where $Z(g) = \{b \in B \colon\ g(b) = 0\}$.

Multivariate elementary functions 3

Exercise: what is $k_1(x)/k_0(x)$? The ratio of two functions is always defined! Multivariate composition works as follows.

Definition (composition)

Let $f(y_1, \ldots, y_m)\colon A \to \mathbb{R}$ with $A \subset \mathbb{R}^m$ and $g_i = g_i(x_1, \ldots, x_n)\colon B_i \to \mathbb{R}$ with $B_i \subset \mathbb{R}^n$ for $i = 1, \ldots, m$. We define the function $f(\overline{g})(x_1, \ldots, x_n) = f(g_1, \ldots, g_m)(x_1, \ldots, x_n)$ with the definition domain

$$D(f(\overline{g})) = \left\{ \overline{a} \in \bigcap_{i=1}^{m} B_i \colon \ \langle g_1(\overline{a}), \ldots, g_m(\overline{a}) \rangle \in A \right\} \quad (\subset \mathbb{R}^n)$$

and values $f(\overline{g})(\overline{a}) := f(g_1(a_1, \ldots, a_n), \ldots, g_m(a_1, \ldots, a_n))$.

So we know the basic functions and projections, and we know the operations of sum, product, and ratio of two functions, as well as the operation of composition of several functions. We proceed to the summarizing definition of multivariate elementary functions.

Multivariate elementary functions 4

The abridged, simplified definition produced by AI is this.

Definition (elementary functions)

A function $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$ is *elementary* if it can be obtained from basic functions and projections by finitely many applications of:

- sum/product/ratio,
- composition.

Equivalently: there is a finite generating word $\langle f_1, \ldots, f_k \rangle$ with $f_k = f$, where each f_i is basic/projection or built from earlier entries by those operations.

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The full detailed definition is this. A *function* in it is any $f : A \to \mathbb{R}$ with $A \subset \mathbb{R}^n$.

Definition (elementary functions)

We call a function f *elementary* if there exists a finite sequence

$$g = \langle f_1, f_2, \ldots, f_k \rangle$$

of $k \in \mathbb{N}$ functions f_i such that $f_k = f$ and for every index $i \in \{1, \ldots, k\}$, the function $f_i = f_i(x_1, \ldots, x_n)$ is one of the basic functions in the above definition or one of the projections, or one of two cases holds.

1. For some two indices $j, j' < i$, we have that $f_i = f_j + f_{j'}$ or $f_i = f_j \cdot f_{j'}$ or $f_i = f_j / f_{j'}$.
2. For some $m + 1$ indices $j, j_1, \ldots, j_m < i$, we have $f_j = f_j(y_1, \ldots, y_m)$, $f_{j_l} = f_{j_l}(x_1, \ldots, x_n)$ for $l = 1, \ldots, m$, and $f_i = f_j(f_{j_1}, \ldots, f_{j_m})$.

We call g the *generating word* of $f = f_k$. Exercise: every f_i is elementary.

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An example of a multivariate elementary function is $f(x,y,z) = \frac{\exp(x+y)}{x^3-z^3}$:

$$f(x,\,y,\,z) = \exp(p_{3,1}(x,\,y,\,z) + p_{3,2}(x,\,y,\,z))/(p_{3,1}(x,\,y,\,z)^3 - p_{3,3}(x,\,y,\,z)^3)$$

where the denominator is in more detail

$$\begin{aligned}&\left((p_{3,1}(x,\,y,\,z)\cdot p_{3,1}(x,\,y,\,z))\cdot p_{3,1}(x,\,y,\,z)\right) + \\ &+ \left(k_{-1}(p_{3,3}(x,\,y,\,z))\cdot((p_{3,3}(x,\,y,\,z)\cdot p_{3,3}(x,\,y,\,z))\cdot p_{3,3}(x,\,y,\,z))\right).\end{aligned}$$

Proposition (continuity of MV elementary functions)

Every multivariate elementary function is continuous.

Proof.

(Sketch) Induction along a generating word $g = \langle f_1, f_2, \ldots, f_k\rangle$ of $f = f_k$ shows that every function f_i is continuous. □

Some more results on continuous maps 1

We first review some results of this kind that we know from the previous lectures. We state them now for metric spaces, although we mostly proved them for topologies.

Let M, N be metric spaces, $X \subset M$, and $f\colon M \to N$ be continuous.

- If X is compact, then $f[X]$ is compact.
- If M is compact, then f is uniformly continuous.
- If X is compact and $N = \mathbb{R}$, then f attains minimum and maximum on X.
- If X is connected, then $f[X]$ is connected.

We also proved, for topological spaces, that if $f\colon T \to U$ is continuous and injective, T is compact, and U is Hausdorff, then the inverse $f^{-1}\colon f[T] \to T$ is continuous. We prove sequentially the metric version.

Proposition

Let M, N be metric spaces, M compact, and let $f\colon M \to N$ be a continuous injection. Then the inverse $f^{-1}\colon f[M] \to M$ is continuous.

Some more results on continuous maps 2

Here is the proof using sequential compactness.

Proof.

Let M be compact, $b \in f[M]$, and let

$$(b_n) \subset f[M] \text{ be a sequence with } \lim b_n = b\,.$$

Let $a := f^{-1}(b)$ and $a_n := f^{-1}(b_n)$ $(\in M)$. We show that $\lim a_n = a$, which by Heine's continuity proves the continuity of f^{-1} at b. We show that every subsequence of (a_n) has a subsequence with the limit a. Exercise below then implies that $\lim a_n = a$. Let (a'_n) be a subsequence of (a_n). We use that M is compact and take a subsequence (a_{m_n}) of (a'_n) with $\lim a_{m_n} = c \in M$. By Heine's continuity we have $\lim f(a_{m_n}) = f(c)$. Since $(f(a_{m_n}))$ is a subsequence of (b_n), we have $f(c) = b$. Since f is injective, $c = a$. □

Exercise: Let $a \in M$ and let $(a_n) \subset M$ be a sequence in a metric space M such that every subsequence of (a_n) has a subsequence with the limit a. Prove that $\lim a_n = a$.

Some more results on continuous maps 3

We conclude with two exercises.

Theorem (continuous inverse)

Let $M = \langle M, d \rangle$ and $N = \langle N, e \rangle$ be metric spaces. Let $f : M \to N$ be an injection and let $c > 0$ be such that

$$\forall x, y \in M: \quad e(f(x), f(y)) \geq c \cdot d(x, y).$$

Then $f^{-1} : f[M] \to M$ is continuous.

Another exercise shows that continuity is not generally preserved in inverses.

- Let $f : \mathbb{N}_0 \to \mathbb{R}$, where $\mathbb{N}_0 \subset \mathbb{R}$, be given by $f(0) = 0$ and $f(n) = \frac{1}{n}$ for $n \in \mathbb{N}$.
- Show: f is continuous and injective.
- Show: f^{-1} is not continuous at $0 \in f[\mathbb{N}_0]$.

Thanks!

THANK YOU FOR YOUR ATTENTION!