Mathematical Analysis 2 (MA2) – Lecture 5

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Complete texts of the lectures are on `https://kam.mff.cuni.cz/~klazar/MAII26en.html` and students are supposed to study them carefully; these slides are just outlines.

This lecture

1. Part 2: Multivariate differential calculus
2. Limits of maps between metric spaces
3. Properties of limits of maps
4. Properties of continuous maps

We are not leaving metric spaces!

We begin Part 2 (multivariate differential calculus). Before we get to derivatives, which will not be in this lecture, we treat the following topics.

- Properties of limits of maps $f : X \to N$, where $X \subset M$, for metric spaces M and N.
- Properties of continuous maps $f : M \to N$.

We are especially interested in how limits and continuity interact with arithmetic operations when $M = \mathbb{R}^n = E_n$ or $N = \mathbb{R}^m = E_m$. Recall that E_n is a complete metric space and that compact sets in E_n are exactly the sets that are closed and bounded.

Center-deleted balls, limit points, proper limits of maps

- For a ball $B_M(b, r)$ in a metric space M, we define the *center-deleted ball* simply by deleting the center of the ball: $C_M(b, r) := B_M(b, r) \setminus \{b\}$.
- A point $b \in M$ is a *limit point* of a set $X \subset M$ if $\forall r > 0$: $C_M(b, r) \cap X \neq \emptyset$.

Definition (proper limit of a map)

Let $M = \langle M, d \rangle$, $N = \langle N, e \rangle$ be metric spaces, $X \subset M$, $f : X \to N$, and $b \in M$ be a limit point of X. We say that

$$\lim_{x \to b} f(x) = c \quad (\in N)$$

if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$f\big[C_M(b, \delta)\big] \subset B_N(c, \varepsilon).$$

Recall that $f[C_M(b, \delta)] = f[C_M(b, \delta) \cap X]$. Why do we assume that b is a limit point of X?

Infinite limits of maps

For $N = \mathbb{R} = E_1$ we allow infinite values of limits.

Definition (infinite limit of a map)

Let $M = \langle M, d \rangle$ be a metric space, $X \subset M$, $f : X \to \mathbb{R}$, and $b \in M$ be a limit point of X. We say that

$$\lim_{x \to b} f(x) = +\infty \quad (\text{resp. } = -\infty)$$

if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$f\big[C_M(b, \delta)\big] \subset (\varepsilon^{-1}, +\infty) \quad (\text{resp. } \subset (-\infty, -\varepsilon^{-1})).$$

In the course *MA1*, we considered besides infinite values of limits also limits at infinities; for example, $\lim_{x \to -\infty} \frac{1}{x} = 0$. For simplicity of presentation (things are already complicated :-)), we will not do this here. Thus if we write $x \to b$, always $b \in M$.

Properties of limits of maps 1: restrictions

The limit of a map is not affected by restricting the map.

Proposition

Let $\langle M, d\rangle$ and $\langle N, e\rangle$ be metric spaces, $X \subset M$, $f\colon X \to N$, $b \in M$ be a limit point of X, and let $Y \subset M$ be such that b is still a limit point of $Y \cap X$. Then

$$\lim_{x\to b} f(x) = C \ (\in \{c, -\infty, +\infty\}) \Rightarrow \lim_{x\to b} (f \mid Y)(x) = C.$$

Proof.

The implication follows from the inclusion $C_M(b, \delta) \cap Y \cap X \subset C_M(b, \delta) \cap X$. We can say more strongly: for a given ε, the same δ works for $f \mid Y$ as for f. □

Exercise: show that the implication cannot, in general, be reversed.

Properties of limits of maps 2: coordinate limits

For $N = \mathbb{R}^m = E_m$, the limit of a map to N can be reduced to limits of coordinate functions.

Proposition

Let $m \in \mathbb{N} \setminus \{1\}$, $M = \langle M, d \rangle$ be a metric space, $X \subset M$, $F = \langle f_1, \ldots, f_m \rangle \colon X \to \mathbb{R}^m$, $b \in M$ be a limit point of X, and let $\overline{c} = \langle c_1, \ldots, c_m \rangle \in \mathbb{R}^m$. Then

$$\lim_{x \to b} F(x) = \overline{c} \iff \forall\, i = 1, \ldots, m \colon \lim_{x \to b} f_i(x) = c_i\,.$$

Proof.

We prove $\Rightarrow$. Let $\lim_{x \to b} F(x) = \overline{c}$, let $i \in \{1, \ldots, m\}$, and let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that if $x \in X$ with $0 < d(x, b) < \delta$, then $\|F(x) - \overline{c}\| < \varepsilon$. So for the same x,

$$|f_i(x) - c_i| \leq \sqrt{\textstyle\sum_{j=1}^{m} (f_j(x) - c_j)^2} = \|F(x) - \overline{c}\| < \varepsilon$$

and we see that $\lim_{x \to b} f_i(x) = c_i$. The opposite implication $\Leftarrow$ is an exercise. □

Limits of $f : \mathbb{R}^2 \to \mathbb{R}$ at $\langle 0, 0 \rangle$ showing that iterated limits do not help

Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $\overline{0} = \langle 0, 0 \rangle$. One might attempt to compute the limit $\lim_{\overline{x} \to \overline{0}} f(\overline{x})$ as the iterated limit $\lim_{x_1 \to 0} \lim_{x_2 \to 0} f(x_1, x_2)$ or $\lim_{x_2 \to 0} \lim_{x_1 \to 0} f(x_1, x_2)$. The following three examples, which are exercises for you, show that this method does not work. The undefined values of f are set to 0.

- $f(x_1, x_2) = \dfrac{x_1 x_2}{x_1^2 + x_2^2}$: both iterated limits are 0, but the 2D limit does not exist.
- $f(x_1, x_2) = \dfrac{x_1^2 - x_2^2}{x_1^2 + x_2^2}$: the two iterated limits disagree (1 vs -1).
- $f(x_1, x_2) = x_1 + x_2 \sin(1/x_1)$: one iterated limit does not exist, the other $=$ the 2D limit $= 0$.

The arithmetic in $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$

In order to deal precisely with infinite limits, we need a precise arithmetic involving infinities. We define the arithmetic in the structure

$$\mathbb{R}^* = \langle \mathbb{R}^*, 0, 1, +, \cdot, /, < \rangle$$

as follows. We extend the usual arithmetic in $\langle \mathbb{R}, 0, 1, +, \cdot, /, < \rangle$ by the following rules.

- $a + (+\infty) = (+\infty) + a = +\infty$ for any $a \in \mathbb{R} \cup \{+\infty\}$,
- $a + (-\infty) = (-\infty) + a = -\infty$ for any $a \in \mathbb{R} \cup \{-\infty\}$,
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \pm\infty$ for any real $a > 0$ and
- $a \cdot (\pm\infty) = (\pm\infty) \cdot a = \mp\infty$ for any real $a < 0$ (all upper or all lower signs),
- $(\pm\infty) \cdot (\pm\infty) = +\infty$ and $(\pm\infty) \cdot (\mp\infty) = -\infty$ (all upper or all lower signs),
- $(\pm\infty)/a = \pm\infty$ for any real $a > 0$ and
- $(\pm\infty)/a = \mp\infty$ for any real $a < 0$ (all upper or all lower signs),
- $a/(\pm\infty) = 0$ for any real a, and, finally,
- $-\infty < +\infty$, $-\infty < a$, and $a < +\infty$ for any real a.
- We change signs by $-(\pm\infty) = \mp\infty$.

Indefinite expressions. Properties of limits of maps 3: Heine

The expressions not mentioned above,

$$(+\infty)+(-\infty),\ (-\infty)+(+\infty),\ 0\cdot(\pm\infty),\ (\pm\infty)\cdot 0,\ \frac{\pm\infty}{\pm\infty},\ \text{ and }\ \frac{A}{0}$$

(any combination of signs and $A \in \mathbb{R}^*$) are indefinite expressions; their values are not defined.

Theorem (limits of maps in Heine's style)

Let M, N, X, f, b, and $C \in \{c, -\infty, +\infty\}$ be as in the two above definitions of limits. Then

$$\lim_{x\to b} f(x) = C \iff \forall\,(a_n) \subset X \setminus \{b\}\colon\ \lim_{n\to\infty} a_n = b \Rightarrow \lim_{n\to\infty} f(a_n) = C\,.$$

Proof.

(Sketch.) The implication $\Rightarrow$ is immediate. In the implication $\neg \Rightarrow \neg$ we define the bad sequence (a_n) by means of AC. □

Arithmetic of limits of real sequences

We rely on the following theorem contained in the course *MA1* (when I was teaching it).

Theorem (arithmetic of limits of real sequences)

Let $(a_n), (b_n) \subset \mathbb{R}$ *be real sequences such that* $\lim_{n\to\infty} a_n = C$ *and* $\lim_{n\to\infty} b_n = D$*, where* $C, D \in \mathbb{R}^*$*. The following holds.*

1. *If* $C + D$ *is not indefinite, then* $\lim_{n\to\infty}(a_n + b_n) = C + D$.
2. *If* CD *is not indefinite, then* $\lim_{n\to\infty} a_n b_n = CD$.
3. *If* C/D *is not indefinite, then* $\lim_{n\to\infty} a_n/b_n = C/D$.

In part 3, we ignore the possibly finitely many undefined ratios a_n/b_n *with* $b_n = 0$.

Exercise: if C/D is not indefinite, then $b_n = 0$ for only finitely many n.
We deduce with the help of the two previous theorems the arithmetic of limits of maps.

Properties of limits of maps 4: arithmetic of limits

Corollary (arithmetic of limits of maps)

Let $M = \langle M, d \rangle$ be a metric space, $X \subset M$, let $f, g \colon X \to \mathbb{R}$, and $b \in M$ be a limit point of X. Let $\lim_{x \to b} f(x) = C$ and $\lim_{x \to b} g(x) = D$, where $C, D \in \mathbb{R}^*$. The following holds.

1. *If $C + D$ is not indefinite, then $\lim_{x \to b}(f + g)(x) = C + D$.*
2. *If CD is not indefinite, then $\lim_{x \to b}(fg)(x) = CD$.*
3. *If C/D is not indefinite, then $\lim_{x \to b}(f/g)(x) = C/D$.*

Proof ideas: We use the two previous theorems.

Proposition (limits of linear combinations)

Let $m \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $M = \langle M, d \rangle$ be a metric space, $X \subset M$, let $F, G \colon X \to \mathbb{R}^m$, and $b \in M$ be a limit point of X. Let $\lim_{x \to b} F(x) = \overline{c}$ and $\lim_{x \to b} G(x) = \overline{d}$. Then $\lim_{x \to b}(\alpha F + \beta G)(x) = \alpha \overline{c} + \beta \overline{d}$.

Properties of limits of maps 5: Cauchy condition

Proof.

(Sketch) For $\alpha, \beta \in \mathbb{R}$ and $u, v \in \mathbb{R}^m$, we use that $\|\alpha u + \beta v\| \leq |\alpha| \cdot \|u\| + |\beta| \cdot \|v\|$. □

Theorem (Cauchy condition)

Let $M = \langle M, d \rangle$ and $N = \langle N, e \rangle$ be metric spaces, with N complete. Let $X \subset M$, $f : X \to N$, and let $b \in M$ be a limit point of X. Then $\lim_{x \to b} f(x)$ exists $\iff$ for every $\varepsilon > 0$ there is $\delta > 0$ and a ball B_N with radius ε such that

$$f[C_M(b, \delta)] \subset B_N\,.$$

Proof.

(Sketch) The implication $\Rightarrow$ is immediate. To prove $\Leftarrow$, we first define $c := \lim f(a_n)$ $(\in N)$ for some sequence $(a_n) \subset X \setminus \{b\}$ with $\lim a_n = b$. Then we show that even $\lim_{x \to b} f(x) = c$. □

Properties of limits of maps 6: monotone maps

The next definition of monotonicity is inspired by quasi-monotone sequences in $\mathbb{R}$.

Definition (f increases/decreases at b)

Let $M = \langle M, d\rangle$ be a metric space, $X \subset M$, $f\colon X \to \mathbb{R}$, and let $b \in M$ be a limit point of X. Then f increases at b if for every $a \in X \setminus \{b\}$ there exists $\delta > 0$ such that

$$\forall\, c \in C_M(b, \delta) \cap X\colon\ f(c) \geq f(a)\,.$$

To define that f decreases at b, we reverse the last inequality.

Theorem (limits of monotone maps)

Let $M = \langle M, d\rangle$ be a metric space, $X \subset M$, $f\colon X \to \mathbb{R}$, and let $b \in M$ be a limit point of X. If f increases (resp. decreases) at b, then

$$\lim_{x \to b} f(x) = \sup(f[X \setminus \{b\}]) \quad (\textit{resp.}\ \ = \inf(f[X \setminus \{b\}])) \quad (\in \mathbb{R}^*).$$

Properties of limits of maps 7: Bolzano–Weierstrass

The supremum and infimum in the previous theorem are taken in the linear order $\langle \mathbb{R}^*, < \rangle$. Show as an exercise that every set in it has infimum and supremum.

Proof.

(Sketch) Similar to the proof for real sequences. □

A sequence $(a_n) \subset M$ is bounded if $(a_n) \subset B_M$ for a ball B_M.

Proposition (simplistic Bolzano–Weierstrass)

Let $\langle M, d \rangle$ be a metric space such that every closed ball in M is compact. Then every bounded sequence $(a_n) \subset M$ has a convergent subsequence (subsequence with a limit).

Proof.

Exercise. □

Exercise: prove that every bounded sequence in the space E_m has a convergent subsequence.

Properties of limits of maps 8: composite limits

For $g\colon A \to B$ and $f\colon A' \to B'$, we set $D(f(g)) = \{a \in A\colon\ g(a) \in A'\}$ (the domain of $f(g)$).

Theorem (composite limits)

Let $\langle M, d\rangle$, $\langle N, e\rangle$, and $\langle P, h\rangle$ be three metric spaces, $X \subset M$, $Y \subset N$, $g\colon X \to N$, $f\colon Y \to P$, $b \in M$ be a limit point of X, $c \in N$ be a limit point of Y, and let b be also a limit point of $D(f(g))$. Suppose that $\lim_{x\to b} g(x) = c$ and $\lim_{y\to c} f(y) = p \in P$. Then the composite map $f(g)$ has limit

$$\lim_{x\to b} f(g)(x) = p$$

if and only if one of two conditions holds.

1. *Either $c \notin Y$, or $c \in Y$ and $f(c) = p$.*
2. *For some $\theta > 0$ we have $c \notin g[C_M(b, \theta)]$.*

If neither condition holds, then $c \in Y$, $f(c) \neq p$, and the limit $\lim_{x\to b} f(g)(x)$ does not exist or equals $f(c)$.

Properties of continuous maps 1: intermediate value theorem

It is worthwhile to read the proof of the previous theorem in the lecture notes.

Theorem (intermediate values)

Let $M = \langle M, d \rangle$ be a connected metric space and let $f\colon M \to \mathbb{R}$ be a continuous map. If $a, b \in M$ and $c \in \mathbb{R}$ are such that

$$f(a) < c < f(b)\,,$$

then there exists a point $e \in M$ such that $f(e) = c$.

Proof.

With what we have all learned, the proof is easy. By Theorem 6 in lecture 4, the image $f[M]$ is a connected subset of $\mathbb{R}$. By Theorem 2 in the same lecture, $f[M]$ is an interval. Since $f(a), f(b) \in f[M]$, also $c \in f[M]$ and we are done. □

Thanks!

THANK YOU FOR YOUR ATTENTION!