MA2 – lecture 5

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Abstract

We begin the second part of the course; it is devoted to multivariate differential calculus. Before we get to derivatives, we treat on the general level of metric spaces limits of maps and their continuity, especially the interplay with arithmetic operations. We specialize spaces to $\mathbb{R}^n$ only when we need arithmetic or order. We obtain the following analogs of univariate results on functions and sequences: the arithmetic of limits, the Cauchy condition, limits of monotone maps, the Bolzano–Weierstrass theorem, limits of composite maps, and the generalization of the intermediate value theorem to connected metric spaces.

Part 2. Multivariate differential calculus

The second part of this course is devoted to multivariate differential calculus. Before we get to the derivatives of functions and maps $f\colon X \to \mathbb{R}$ and $f\colon X \to \mathbb{R}^m$, where $X \subset \mathbb{R}^n$, we will discuss the limits of maps and their continuity at a general level of metric spaces. We are especially interested in the interplay with arithmetic operations. If we need these operations or ordering, we specialize spaces to $\mathbb{R}^n$. Recall that $\mathbb{R}^n = E_n$ is a complete metric space with the metric e_n. In E_n, compact sets are exactly the closed and bounded sets. E_n is a normed space, even an inner product space, and we shall write the metric $e_n(\overline{x}, \overline{y})$ as $\|\overline{x} - \overline{y}\|$, where the Euclidean norm $\|\cdot\|$ is given by

$$\|\overline{x}\| = \|\langle x_1, x_2, \ldots, x_n\rangle\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}\,.$$

Limits of maps between metric spaces

For any ball $B(b,r)$ in a metric space M, we define the center-deleted ball $C(b,r)$ by

$$C_M(b,\,r) = C(b,\,r) := B(b,\,r) \setminus \{b\} \quad (\subset M)\,.$$

Let $X \subset M$ and $b \in M$. The point b is a *limit point* of the set X if

$$\forall\, r > 0\colon\ C(b,\,r) \cap X \neq \emptyset\,.$$

Exercise 1 *Show that b is a limit point of X if and only if there is a sequence $(a_n) \subset X \setminus \{b\}$ such that $\lim_{n\to\infty} a_n = b$.*

Definition 2 (limits of maps) *Let $M = \langle M, d\rangle$ and $N = \langle N, e\rangle$ be metric spaces, $X \subset M$, $f\colon X \to N$, $b \in M$ be a limit point of X, and let $c \in N$. If for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$f[C_M(b,\,\delta)] \subset B_N(c,\,\varepsilon)\,,$$

we write $\lim_{x\to b} f(x) = c$ or $f(x) \to c$ as $x \to b$, and say that f has limit c at the point b.

Recall that the image $f[C_M(b,\delta)] = f[C_M(b,\delta) \cap X]$. As we remember from *MA1*, without the assumption that b is a limit point of X, the definition of limit does not make good sense.

Exercise 3 *Why?*

In other words, $\lim_{x\to b} f(x) = c$ says that for every $\varepsilon > 0$ there is $\delta > 0$ such that if $a \in X$ with $0 < d(a,b) < \delta$, then $e(f(a),c) < \varepsilon$.

If $N = E_1$, we allow limits $\pm\infty$. We define $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$. We review the arithmetic in $\mathbb{R}^*$ later.

Definition 4 (infinite limits) *Let $M = \langle M, d\rangle$ be a metric space, $X \subset M$, $f\colon X \to \mathbb{R}$, and let $b \in M$ be a limit point of X. If for every $\varepsilon > 0$ there is $\delta > 0$ such that*

$$f[C_M(b,\,\delta)] \subset (1/\varepsilon,\,+\infty) =: B_{\mathbb{R}}(+\infty,\,\varepsilon)\,,$$

we write $\lim_{x\to b} f(x) = +\infty$ or $f(x) \to +\infty$ as $x \to b$, and say that f has limit $+\infty$ at the point b. To define that f has limit $-\infty$ at b, we replace the last interval with $(-\infty, -1/\varepsilon) := B_{\mathbb{R}}(-\infty, \varepsilon)$.

In *MA1*, $M = N = E_1$ and we also considered limits $\lim_{x\to\pm\infty} f(x)$ at infinities. For example, $\lim_{x\to-\infty} \frac{1}{x} = 0$. We do not allow them here; if we write $x \to b$, then always $b \in M$, so that b is an element of the space.

Exercise 5 *Restate Definitions 2 and 4 in terms of norms if $M = E_n$ or $N = E_m$.*

We show that limits are preserved under restrictions. Recall that if $f\colon A \to B$ is a map and C is any set, then the restriction $f \,|\, C\colon A \cap C \to B$ has values $(f \,|\, C)(x) = f(x)$.

Proposition 6 *Let M, N, X, f, b, and $C \in \{c, -\infty, +\infty\}$ be as in Definitions 2 and 4. Let $Y \subset M$ be such that b is still a limit point of $Y \cap X$. Then*

$$\lim_{x\to b} f(x) = C \Rightarrow \lim_{x\to b} (f \,|\, Y)(x) = C\,.$$

Proof. We could state the implication even more strongly: for the restriction $f \,|\, Y$ and given $\varepsilon > 0$, the same $\delta > 0$ works as for the map f. This follows from the inclusion

$$C_M(b,\, \delta) \cap Y \cap X \subset C_M(b,\, \delta) \cap X\,.$$

□

Exercise 7 *Can the implication in the proposition be reversed?*

We show that if $N = E_m$, then the existence of limits of maps is equivalent to the existence of limits of all their coordinate functions.

Proposition 8 *Let $m \in \mathbb{N}$ with $m \geq 2$. Suppose that $M = \langle M, d\rangle$ is a metric space, $X \subset M$, that $F = \langle f_1, \ldots, f_m\rangle\colon X \to \mathbb{R}^m$, that $b \in M$ is a limit point of X, and that $\overline{c} = \langle c_1, \ldots, c_m\rangle \in \mathbb{R}^m$. Then*

$$\lim_{x\to b} F(x) = \overline{c} \iff \forall\, i = 1,\, \ldots,\, m\colon\ \lim_{x\to b} f_i(x) = c_i\,.$$

Proof. Let $\lim_{x\to b} F(x) = \overline{c}$, let $i \in \{1, \ldots, m\}$, and let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that if $x \in X$ with $0 < d(x, b) < \delta$, then $\|F(x) - \overline{c}\| < \varepsilon$. Thus for the same points x,

$$|f_i(x) - c_i| \leq \sqrt{\textstyle\sum_{j=1}^{m} (f_j(x) - c_j)^2} = \|F(x) - \overline{c}\| < \varepsilon$$

and we see that $\lim_{x\to b} f_i(x) = c_i$.

Suppose that $\lim_{x\to b} f_i(x) = c_i$ for every $i \in \{1, \ldots, m\}$ and that $\varepsilon > 0$ is given. Then there exists $\delta > 0$ such that for every $i \in \{1, \ldots, m\}$ and every $x \in X$ with $0 < d(x, b) < \delta$ we have $|f_i(x) - c_i| < \varepsilon$. Thus for the same x,

$$\|F(x) - \overline{c}\| = \sqrt{\textstyle\sum_{j=1}^{m} (f_j(x) - c_j)^2} < \sqrt{\textstyle\sum_{j=1}^{m} \varepsilon^2} = \sqrt{m} \cdot \varepsilon$$

and we see that $\lim_{x\to b} F(x) = \overline{c}$. □

For $m = 1$, the proposition holds trivially and also with infinite limits $\pm\infty$.

Let $M = E_n$. Then we could try to find the limits

$$\lim_{\overline{x}\to\overline{b}} f(x_1, \ldots, x_n)$$

of coordinate functions $f\colon X \to \mathbb{R}$, where $X \subset \mathbb{R}^n$, by successively (in some order $x_{i_n}, \ldots, x_{i_1}$) computing for all n variables $x_1, \ldots, x_n$ the so-called *iterated limit*

$$\lim_{x_{i_1}\to b_{i_1}} \ldots \lim_{x_{i_n}\to b_{i_n}} f(x_1, \ldots, x_n)$$

(the computation starts with the inner limit $\lim_{x_{i_n}\to b_{i_n}}$, and the values of the $n - 1$ variables with respect to which we are not taking a limit are always kept fixed). The next three exercises for functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ and the point $\overline{b} = \overline{0} = \langle 0, 0\rangle$ show that this idea does not work.

Exercise 9 *Let $f(x_1, x_2) = x_1 x_2/(x_1^2 + x_2^2)$ for $x_1^2 + x_2^2 \neq 0$ and $f(0, 0) = 0$. Then the limit $\lim_{\overline{x}\to\overline{0}} f(x_1, x_2)$ does not exist, even if*

$$\lim_{x_1\to 0} \lim_{x_2\to 0} f(x_1, x_2) = \lim_{x_2\to 0} \lim_{x_1\to 0} f(x_1, x_2) = 0\,.$$

Exercise 10 *Let $f(x_1, x_2) = (x_1^2 - x_2^2)/(x_1^2 + x_2^2)$ for $x_1^2 + x_2^2 \neq 0$ and $f(0, 0) = 0$. Then*

$$\lim_{x_1\to 0} \lim_{x_2\to 0} f(x_1, x_2) = 1 \quad \textit{and} \quad \lim_{x_2\to 0} \lim_{x_1\to 0} f(x_1, x_2) = -1\,.$$

Exercise 11 *Let $f(x_1, x_2) = x_1 + x_2 \sin(1/x_1)$ for $x_1 \neq 0$ and $f(0, x_2) = 0$. Then*

$$\lim_{\overline{x}\to\overline{0}} f(\overline{x}) = \lim_{x_1\to 0} \lim_{x_2\to 0} f(x_1, x_2) = 0\,,$$

but $\lim_{x_2\to 0} \lim_{x_1\to 0} f(x_1, x_2)$ does not exist.

These exercises are taken from [1, pp. 421–422].

Properties of limits of maps

We begin with the arithmetic of limits. We recall that if $f, g\colon A \to \mathbb{R}$ are any functions, then the sum function $f+g\colon A \to \mathbb{R}$ has values $(f+g)(a) = f(a)+g(a)$. The product function fg and the ratio function f/g are defined similarly. We review the arithmetic in the structure

$$\mathbb{R}^* = \langle \mathbb{R}^*,\, 0,\, 1,\, +,\, \cdot,\, /,\, < \rangle \quad (\mathbb{R}^* = \mathbb{R} \cup \{-\infty,\, +\infty\})\,.$$

We extend the usual arithmetic in $\langle \mathbb{R}, 0, 1, +, \cdot, /, < \rangle$ by the rules $a+(+\infty) = (+\infty)+a = +\infty$ for any $a \in \mathbb{R} \cup \{+\infty\}$, $a+(-\infty) = (-\infty)+a = -\infty$ for any $a \in \mathbb{R} \cup \{-\infty\}$, $a \cdot (\pm\infty) = (\pm\infty)\cdot a = \pm\infty$ for any real $a > 0$ and $a \cdot (\pm\infty) = (\pm\infty)\cdot a = \mp\infty$ for any real $a < 0$ (all upper or all lower signs), $(\pm\infty)\cdot(\pm\infty) = +\infty$ and $(\pm\infty)\cdot(\mp\infty) = -\infty$ (all upper or all lower signs), $(\pm\infty)/a = \pm\infty$ for any real $a > 0$ and $(\pm\infty)/a = \mp\infty$ for any real $a < 0$ (all upper or all lower signs), $a/(\pm\infty) = 0$ for any real a, and, finally, $-\infty < +\infty$, $-\infty < a$, and $a < +\infty$ for any real a; we change the sign by $-(\pm\infty) = \mp\infty$. The not mentioned expressions

$$(+\infty)+(-\infty),\ (-\infty)+(+\infty),\ 0\cdot(\pm\infty),\ (\pm\infty)\cdot 0,\ \frac{\pm\infty}{\pm\infty},\ \text{ and }\ \frac{A}{0}$$

(any combination of signs and $A \in \mathbb{R}^*$) are called indefinite expressions. Their values are not defined.

Exercise 12 *Compute* $(+\infty)-(-\infty)$, $\frac{-\infty}{-6}$, *and* $(-\infty)\cdot(-\infty)$.

Exercise 13 *Show that* $\langle \mathbb{R}^*, < \rangle$ *is a linear order and that every set* $X \subset \mathbb{R}^*$ *has infimum and supremum in it.*

We deduce the arithmetic of limits of maps from the arithmetic of limits of sequences. We use the Heine-style restatement of Definitions 2 and 4.

Theorem 14 (Heine's limits) *Let* M, N, X, f, b, *and* $C \in \{c, -\infty, +\infty\}$ *be as in Definitions 2 and 4. Then*

$$\lim_{x\to b} f(x) = C \iff \forall\, (a_n) \subset X \setminus \{b\}\colon\ \lim_{n\to\infty} a_n = b \Rightarrow \lim_{n\to\infty} f(a_n) = C\,.$$

Proof. Let $\lim_{x\to b} f(x) = C$, let $C = c \in N$, let $(a_n) \subset X \setminus \{b\}$ have $\lim a_n = b$, and let $\varepsilon > 0$ be given. Since $f(x) \to c$ as $x \to b$, there is $\delta > 0$ such that $a \in C_M(b,\delta) \cap X \Rightarrow f(a) \in B_N(c,\varepsilon)$. Since $a_n \to b$ as $n \to \infty$ and always $a_n \neq b$, there is n_0 such that if $n \geq n_0$, then $a_n \in C_M(b,\delta) \cap X$. Thus for the same indices n we have $f(a_n) \in B_n(c,\varepsilon)$. We see that $\lim f(a_n) = C$.

If $C \in \{-\infty, +\infty\}$ and $N = E_1$, the same argument works, only the ball $B_N(c,\varepsilon)$ is replaced with the "ball" $B_{\mathbb{R}}(C,\varepsilon)$.

Suppose that it is not true that $\lim_{x\to b} f(x) = C$. Then there exists $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there exist — by the axiom of choice — a point $a_n \in C_M(b,n^{-1}) \cap X$ such that $f(a_n)$ is not in $B_N(c,\varepsilon)$ if $C = c$, resp. not in $B_{\mathbb{R}}(C,\varepsilon)$ if $C \in \{+\infty,-\infty\}$. So we obtain a sequence $(a_n) \subset X \setminus \{b\}$ such that $\lim a_n = b$, but it is not true that $\lim f(a_n) = C$.

□

We build on the following theorem on limits of sequences proven in the course *MA1*.

Theorem 15 (arithmetic of limits in MA1) *Let $(a_n), (b_n) \subset \mathbb{R}$ be real sequences such that $\lim_{n\to\infty} a_n = C$ and $\lim_{n\to\infty} b_n = D$, where $C, D \in \mathbb{R}^*$. The following holds.*

1. *If $C + D$ is not indefinite, then $\lim_{n\to\infty}(a_n + b_n) = C + D$.*
2. *If CD is not indefinite, then $\lim_{n\to\infty} a_n b_n = CD$.*
3. *If C/D is not indefinite, then $\lim_{n\to\infty} a_n/b_n = C/D$.*

In part 3, we ignore the possibly finitely many undefined ratios a_n/b_n with $b_n = 0$.

Indeed, if C/D is not indefinite, then $D \neq 0$ and since $\lim b_n = D \neq 0$, we have $b_n = 0$ for only finitely many n.

Corollary 16 *Let $M = \langle M, d\rangle$ be a metric space, $X \subset M$, let $f, g\colon X \to \mathbb{R}$, and $b \in M$ be a limit point of X. Let $\lim_{x\to b} f(x) = C$ and $\lim_{x\to b} g(x) = D$, where $C, D \in \mathbb{R}^*$. Then the following holds.*

1. *If $C + D$ is not indefinite, then $\lim_{x\to b}(f+g)(x) = C + D$.*
2. *If CD is not indefinite, then $\lim_{x\to b}(fg)(x) = CD$.*

3. If C/D is not indefinite, then $\lim_{x\to b}(f/g)(x) = C/D$.

Proof. We only prove part 3 and leave the proofs of parts 1 and 2 as an exercise. We assume that $\lim_{x\to b} f(x) = C$, $\lim_{x\to b} g(x) = D$, and that C/D is not an indefinite expression. By Theorem 14 it suffices to show that whenever $(a_n) \subset M \setminus \{b\}$ is a sequence such that $\lim a_n = b$, then $\lim(f/g)(a_n) = \lim f(a_n)/g(a_n) = C/D$. Let (a_n) be as stated. It follows from the assumptions and Theorem 14 that $\lim f(a_n) = C$ and $\lim g(a_n) = D$. Now by part 3 in Theorem 15 we have that $\lim f(a_n)/g(a_n) = C/D$. □

A subtle point in the previous proof of part 3 is that, unlike for parts 1 and 2, the definition domain $Y \subset X$ of the ratio function $(f/g)(x)$ may be a proper subset of X because the points $a \in X$ with $g(a) = 0$ have to be deleted.

Exercise 17 *Show that if C/D is not indefinite, then b is still a limit point of the definition domain Y of f/g.*

Exercise 18 *Prove parts 1 and 2 of the corollary.*

As we know from the course *MA1*, the theorem on the arithmetic of limits does not exhaust all possible results on the interplay of limits and arithmetic. We leave one such result that is not covered by the corollary as an exercise.

Exercise 19 *Let $M = \langle M, d\rangle$ be a metric space, $X \subset M$, let $f, g\colon X \to \mathbb{R}$, and $b \in M$ be a limit point of X. Let $\lim_{x\to b} f(x) = 0$ and let g be bounded on a center-deleted ball $C_M(b, r)$ $(\cap X)$. Then*

$$\lim_{x\to b}(fg)(x) = 0\,.$$

While $\mathbb{R}$ is a normed field, $\mathbb{R}^m$ for $m > 1$ is only a normed real vector space. We can still prove that limits preserve linear combinations.

Proposition 20 *Let $m \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$, $M = \langle M, d\rangle$ be a metric space, $X \subset M$, let $F, G\colon X \to \mathbb{R}^m$, and $b \in M$ be a limit point of X. Let $\lim_{x\to b} F(x) = \overline{c}$ and $\lim_{x\to b} G(x) = \overline{d}$. Then*

$$\lim_{x\to b}(\alpha F + \beta G)(x) = \alpha\overline{c} + \beta\overline{d}\,.$$

Proof. Let $\varepsilon > 0$ be given. By the assumptions, there is $\delta > 0$ such that $a \in C_M(b, \delta) \cap X \Rightarrow \|F(a) - \overline{c}\|, \|G(a) - \overline{d}\| < \varepsilon$. Thus for the same points a we have

$$\begin{aligned}&\|(\alpha F + \beta G)(a) - (\alpha\overline{c} + \beta\overline{d})\| = \|\alpha(F(a) - \overline{c}) + \beta(G(a) - \overline{d})\| \\ &\leq |\alpha| \cdot \|F(a) - \overline{c}\| + |\beta| \cdot \|G(a) - \overline{d}\| \leq \varepsilon(|\alpha| + |\beta|)\,.\end{aligned}$$

We see that the displayed limit holds. □

Exercise 21 *Explain the first displayed inequality in the proof. Why we cannot write the last inequality as* $< \varepsilon(|\alpha| + |\beta|)$*?*

We obtain two existential results on limits of maps.

Theorem 22 (Cauchy condition) *Let* $M = \langle M, d\rangle$ *and* $N = \langle N, e\rangle$ *be metric spaces, with* N *complete. Let* $X \subset M$, $f\colon X \to N$, *and let* $b \in M$ *be a limit point of* X. *Then* $\lim_{x\to b} f(x)$ *exists* $\iff$ *for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *and a ball* B_N *with radius* ε *such that*

$$f[C_M(b,\, \delta)] \subset B_N\,.$$

Proof. Let $\lim_{x\to b} f(x) = c$ for some $c \in N$ and let $\varepsilon > 0$ be given. We set $B_N = B_N(c, \varepsilon)$. By the definition of limits, there is $\delta > 0$ such that $f[C_M(b,\, \delta)] \subset B_N$.

We assume that the right-hand side of the equivalence holds. We take any sequence $(a_n) \subset X \setminus \{b\}$ such that $\lim a_n = b$. It exists by Exercise 1 because b is a limit point of X. The right-hand side condition implies that the sequence $(f(a_n)) \subset N$ is Cauchy. We use that N is complete and take the limit $c = \lim f(a_n)$ $(\in N)$. We show that $\lim_{x\to b} f(x) = c$. Let $\varepsilon > 0$ be given. We take the $\delta > 0$ and the ball $B_N = B_N(c', \varepsilon/4)$ guaranteed by the right-hand side condition for $\varepsilon/4$. Let $B'_N = B_N(c', \varepsilon/2)$. We claim that $c \in B'_N$. If $c \notin B'_N$, then $e(c, c') \geq \varepsilon/2$. We can take sufficiently large index n such that $d(b, a_n) < \delta$ and $e(c, f(a_n)) < \varepsilon/4$. By TI, $f(a_n) \notin B_N$, which contradicts the definition of B_N. Thus $c \in B_N(c', \varepsilon/2)$. Then for every $a \in C_M(b, \delta) \cap X$ we have that

$$e(f(a),\, c) \leq e(f(a),\, c') + e(c',\, c) < \varepsilon/4 + \varepsilon/2 < \varepsilon\,,$$

and we see that $\lim_{x\to b} f(x) = c$. □

If $M = \langle M, d\rangle$ is a metric space, $X \subset M$, and $f\colon X \to \mathbb{R}$, when is f monotone? The next definition is inspired by quasi-monotone real sequences.

Definition 23 (monotonicity) *Let $M = \langle M, d\rangle$ be a metric space, $X \subset M$, $f\colon X \to \mathbb{R}$, and let $b \in M$ be a limit point of X. Then f increases at b if for every $a \in X \setminus \{b\}$ there exists $\delta > 0$ such that*

$$\forall\, c \in C_M(b,\, \delta) \cap X\colon\ f(c) \geq f(a)\,.$$

To define that f decreases at b, we reverse the last inequality.

Theorem 24 (limits of monotone maps) *Let $M = \langle M, d\rangle$ be a metric space, $X \subset M$, $f\colon X \to \mathbb{R}$, and let $b \in M$ be a limit point of X. If f increases at b, then*

$$\lim_{x\to b} f(x) = \sup(f[X \setminus \{b\}]) \quad (\in \mathbb{R}^*)\,.$$

If f decreases at b, then

$$\lim_{x\to b} f(x) = \inf(f[X \setminus \{b\}]) \quad (\in \mathbb{R}^*)\,.$$

The supremum and infimum are taken in the linear order $\langle \mathbb{R}^*, <\rangle$ (see Exercise 13).

Proof. We suppose that f increases at b; the proof for decreasing f is similar. Let $\varepsilon > 0$ be given and let $A = \sup(f[X \setminus \{b\}])$; then $A \in \mathbb{R}$ or $A = +\infty$. By the definition of supremum, there exists $a \in X \setminus \{b\}$ such that $f(a) \in B_{\mathbb{R}}(A, \varepsilon)$. Since f increases at b and $f(a') \leq A$ for every $a' \in X \setminus \{b\}$, for some $\delta > 0$ we have

$$\forall c \in C_M(b,\, \delta) \cap X\colon\ f(c) \geq f(a),\ \text{ so that }\ f[C_M(b,\, \delta)] \subset U\,.$$

We see that $\lim_{x\to b} f(x) = A$. □

We consider the following generalization of the *Bolzano–Weierstrass theorem*. A sequence $(a_n) \subset M$ in a metric space M is bounded if there is a ball B in M such that $(a_n) \subset B$.

Proposition 25 (metric Bolzano–Weierstrass) *Let $\langle M, d\rangle$ be a metric space such that every closed ball in M is compact. Then every bounded sequence $(a_n) \subset M$ has a convergent subsequence (subsequence with a limit).*

Proof. Let $(a_n) \subset B = B(b,r)$ be a bounded sequence in B. Then also $(a_n) \subset \overline{B} = \overline{B}(b,r)$, and (a_n) has a convergent subsequence due to the assumption on M. □

Exercise 26 *Prove that every bounded sequence in the space E_m has a convergent subsequence.*

We arrive at the last result on limits of maps between metric spaces that we mention, at the *theorem on limits of composite maps*. We recall the generalized composition of maps. If $g\colon A \to B$ and $f\colon A' \to B'$ are two maps, where A, B, A', and B' are arbitrary sets (we do not require that $B = A'$, hence "generalized"), then the composite map $f(g)\colon D(f(g)) \to B'$ has the definition domain

$$D(f(g)) := \{a \in A\colon\ g(a) \in A'\}\ \ (\subset A)$$

and values $f(g)(a) := f(g(a))$. This is the kind of functional composition that is used in mathematical analysis.

Theorem 27 (composite limits) [1] *Let $\langle M,d\rangle$, $\langle N,e\rangle$, and $\langle P,h\rangle$ be three metric spaces, $X \subset M$, $Y \subset N$, $g\colon X \to N$, $f\colon Y \to P$, $b \in M$ be a limit point of X, $c \in N$ be a limit point of Y, and let b be also a limit point of $D(f(g))$. Suppose that $\lim_{x\to b} g(x) = c$ and $\lim_{y\to c} f(y) = p \in P$. Then the composite map $f(g)$ has limit*

$$\lim_{x\to b} f(g)(x) = p$$

if and only if one of two conditions holds.

1. *Either $c \notin Y$, or $c \in Y$ and $f(c) = p$.*
2. *For some $\theta > 0$ we have $c \notin g[C_M(b,\theta)]$.*

If neither condition holds, then $c \in Y$, $f(c) \neq p$, and the limit $\lim_{x\to b} f(g)(x)$ does not exist or equals $f(c)$ (which is not p).

Proof. Let $\varepsilon > 0$ be given. By the assumed limits,

[1] In this precise form, the theorem is due to the author of the lecture notes.

- there is $\delta' > 0$ such that (a) $f[C_N(c, \delta')] \subset B_P(p, \varepsilon)$, and
- there is $\delta > 0$ such that (b) $g[C_M(b, \delta)] \subset B_N(c, \delta')$.

Suppose that condition 1 holds. Then we can strengthen inclusion (a) to the inclusion $f[B_N(c, \delta')] \subset B_P(p, \varepsilon)$, and

$$f(g)[C_M(b,\, \delta)] = f[\, g[C_M(b,\, \delta)]\,] \subset f[B_N(c,\, \delta')] \subset B_P(p,\, \varepsilon)\,.$$

Hence $\lim_{x\to b} f(g)(x) = p$. Suppose that condition 2 holds. We can take the δ such that, additionally, $\delta \leq \theta$, with θ as in condition 2. Then we can strengthen inclusion (b) to $g[C_M(b, \delta)] \subset C_N(c, \delta')$, and

$$f(g)[C_M(b,\, \delta)] = f[\, g[C_M(b,\, \delta)]\,] \subset f[C_N(c,\, \delta')] \subset B_P(p,\, \varepsilon)$$

(Exercise 28). Again, $\lim_{x\to b} f(g)(x) = p$.

Suppose that neither condition holds. Then $c \in Y$, $f(c) \neq p$, and (by AC) for every $n \in \mathbb{N}$ there is a point $a_n \in C_M(b, n^{-1}) \cap X$ such that $g(a_n) = c$. Then $(a_n) \subset D(f(g)) \setminus \{b\}$, $\lim a_n = b$, and

$$\lim\, f(g)(a_n) = \lim\, f(g(a_n)) = \lim\, f(c) = f(c) \quad (\neq p)\,.$$

By Theorem 14, the limit $\lim_{x\to b} f(g(x))$ either does not exist or equals $f(c)$ (which is not p). □

Exercise 28 *Explain the last four displayed inclusions in the proof.*

Properties of continuous maps

We generalize several results from the course *MA1* to continuous maps between metric spaces. We already obtained several such generalizations in the previous lectures (minimax theorem, uniform continuity of continuous maps defined on compact sets, ...). Still some univariate results remain which can be generalized.

Let $\langle M, d\rangle$ and $\langle N, e\rangle$ be metric spaces, and let $f\colon M \to N$. Recall from lecture 1 that f is continuous at a point $b \in M$, if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$f[B_M(b,\, \delta)] \subset B_N(f(b),\, \varepsilon)\,.$$

Equivalently, in Heine' style, for every sequence $(a_n) \subset M$ with $a_n \to b$, also $f(a_n) \to f(b)$ (lecture 1). The map f is continuous if it is continuous at every

point in M. Equivalently, for every open set $B \subset N$, the preimage $f^{-1}[B]$ is an open set in M (lecture 1).

We generalize the *MA1* theorem on attaining intermediate values by a continuous function.

Theorem 29 (intermediate values) *Let $M = \langle M, d\rangle$ be a connected metric space and let $f\colon M \to \mathbb{R}$ be a continuous map. If $a, b \in M$ and $c \in \mathbb{R}$ are such that*

$$f(a) < c < f(b)\,,$$

then there exists a point $e \in M$ such that $f(e) = c$.

Proof. By Theorem 6 in lecture 4, the image $f[M]$ is a connected subset of $\mathbb{R}$. By Theorem 2 in the same lecture, $f[M]$ is an interval. Since $f(a), f(b)$ are in $f[M]$, also $c \in f[M]$ and we are done. □

References

[1] V. A. Zorich, *Mathematical Analysis.* I, Springer-Verlag, Berlin 2004