MA2 − lecture 2

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Abstract

We mention, without proofs, Banach's theorem and Baire's theorem on complete spaces. We prove that products of complete spaces are complete. We non-standardly define completions of metric spaces and prove their existence and uniqueness. We introduce topologies and prove that every metric topology is normal. We review Hilbert spaces and normed vector spaces. We prove the general form of the Cauchy–Schwarz inequality. Finally, we define compact metric spaces.

In the second lecture, we say more about complete metric spaces, and then we turn to other mathematical structures related to metric spaces. In the end, we define compact metric spaces.

Two theorems without proofs on complete spaces

We mention two important theorems on complete metric spaces. We omit their proofs because both theorems and their applications are topics of the course *MA3*.

Let $M = \langle M, d\rangle$ be a metric space and $f\colon M \to M$ be a map. We say that $(a_n) \subset M$ is a *sequence of iterations of* f if

$$a_{n+1} = f(a_n) \text{ for every } n \in \mathbb{N}\,.$$

We call the map f *contractive* if there is a constant $c \in [0, 1)$ such that

$$d(f(x),\, f(y)) \le c \cdot d(x,\, y) \text{ for every } x,\, y \in M\,.$$

Exercise 1 *Show that contractive maps are continuous.*

Exercise 2 *Composition of two contractive maps is contractive.*

Exercise 3 *Is the map* $f_1(x) = x^2 \colon [0,1] \to [0,1]$ *contractive with respect to the metric* e_1*? What about the maps* $f_2(x) = x$ *and* $f_3(x) = \sqrt{x}$*?*

A useful theorem, due to the Polish mathematician *Stefan Banach (1892–1945)*, asserts that for every complete space M and every contractive map $f \colon M \to M$, the equation $f(x) = x$ has a unique solution.

Theorem 4 (Banach's) *Let* $M = \langle M, d \rangle$ *be a complete metric space and let* $f \colon M \to M$ *be a contractive map. Then there exists a unique element* $a \in M$ *such that* $f(a) = a$*. Moreover, for every sequence* (a_n) *of iterations of* f *we have*

$$\lim_{n \to \infty} a_n = a\,.$$

Solutions of the equation $f(x) = x$ are called *fixed points of* f. A basic existence theorem on solutions of differential equations is easily proven with the help of Banach's theorem.

Exercise 5 *Show that for any contractive map* f*, the solution of the equation* $f(x) = x$*, if it exists, is unique.*

The next result on fixed points does not follow from Banach's theorem.

Exercise 6 *Prove that every continuous real function* $f \colon [0,1] \to [0,1]$ *has a fixed point. Hint: consider the function* $f(x) - x$*.*

We introduce sparse sets.

Definition 7 (sparse sets) *Let* $M = \langle M, d \rangle$ *be a metric space and* $X \subset M$*. We say that the set* X *is* sparse (in M) *if for every ball* B *in* M *there exists a ball* B' *in* M *such that* $B' \subset B$ *and* $B' \cap X = \emptyset$*.*

Exercise 8 *Show that any finite union of sparse sets is a sparse set.*

The second theorem on complete spaces that we give without proof is due to the French mathematician *René-Louis Baire (1874–1932)*.

Theorem 9 (Baire's) *Let $M = \langle M, d\rangle$ be a complete metric space. Then in every countable union*

$$M = \bigcup_{n=1}^{\infty} X_n\ ,$$

at least one set X_n is not sparse.

In other words, it is not possible to express a complete space as a countable union of sparse sets.

Exercise 10 *Deduce from Baire's theorem that it is not possible to express any complete space as a finite union of sparse sets.*

We prove an interesting corollary of Baire's theorem. A point $b \in M$ in a metric space M is *isolated* if the ball $B_M(b, r) = \{b\}$ for some radius r.

Exercise 11 *Every point in every discrete metric space is isolated. No point in any Euclidean space E_n is isolated.*

Exercise 12 *A point $b \in M$ in a metric space M is isolated if and only if the set $\{b\}$ is not sparse.*

Corollary 13 *Suppose that $M = \langle M, d\rangle$ is a complete metric space such that no point $b \in M$ is isolated. Then the set M is uncountable.*

Proof. Consider the union

$$M = \bigcup_{b\in M}\{b\}\ .$$

By the assumption and Exercise 12, every set $\{b\}$ in the union is sparse. By the assumption, Theorem 9, and Exercise 10, the union cannot be at most countable. Therefore, the set M is uncountable. $\square$

In the course *MA3*, we prove with the help of Baire's theorem that there is a continuous function $f\colon [0, 1] \to \mathbb{R}$ that is not differentiable at any point $b \in [0, 1]$.

Products of complete metric spaces

By Proposition 31 in lecture 1, a subspace of a complete space M is complete if and only if its base set is a closed subset of M. Now we treat products of spaces.

Proposition 14 *Let $M = \langle M, d \rangle$ and $N = \langle N, e \rangle$ be complete metric spaces. Then the product metric space $P = \langle M \times N, d' \rangle$ is complete.*

Proof. Let $(\overline{c}_n) = (\langle a_n, b_n \rangle)$ be a Cauchy sequence in P. Since

$$d(a_m,\, a_n),\; e(b_m,\, b_n) \le d'(\overline{c}_m,\, \overline{c}_n)\,,$$

the sequences (a_n) and (b_n) are Cauchy sequences in the spaces M and N, respectively. Let $a := \lim a_n$ $(\in M)$, $b := \lim b_n$ $(\in N)$, and $\overline{c} := \langle a, b \rangle$. Since

$$d'(\langle x,\, y \rangle,\, \langle x',\, y' \rangle) \le d(x,\, x') + e(y,\, y')$$

for every $x, x' \in M$ and $y, y' \in N$, by Exercise 15 we have $\lim \overline{c}_n = \overline{c}$. $\square$

Exercise 15 *Explain in detail the last sentence of the previous proof.*

Starting from the next theorem and using the previous proposition, we deduce the completeness of the Euclidean spaces E_n.

Theorem 16 (from MA1) *The Euclidean space $E_1 = \langle \mathbb{R}, |x - y| \rangle$ is complete.*

Corollary 17 *For every $n \in \mathbb{N}$, the Euclidean space E_n is complete.*

Proof. This follows from Proposition 11 in lecture 1, and from Proposition 14 and Theorem 16 in this lecture. $\square$

Completions of metric spaces

This construction generalizes the construction of the field $\mathbb{R}$ of real numbers from the field $\mathbb{Q}$ of fractions. We begin with the definition of dense sets.

Definition 18 (dense sets) *Let $M = \langle M, d \rangle$ be a metric space and $X \subset M$. We call the set X* dense *(in M) if $B \cap X \neq \emptyset$ for every ball B in M.*

Exercise 19 *A set $X \subset M$ is dense $\iff$ for every point $b \in M \setminus X$ there is a sequence $(a_n) \subset X$ such that $\lim a_n = b$ $\iff$ such sequences exist for every $b \in M$.*

We define completions of metric spaces.

Definition 20 (completions) *Let $M = \langle M, d\rangle$ be a metric space. A metric space $C = \langle C, e\rangle$ is a* completion of M *if M and C have three properties.*

1. *M is a subspace of C.*
2. *Every Cauchy sequence $(a_n) \subset M$ has a limit in C.*
3. *The set $C \setminus M$ is inclusion-wise minimal with respect to property 2.*

Our definition is non-standard — we do not require C to be complete, nor that M be dense in C. These properties of C follow as corollaries of our definition, which is logically more natural than the standard definition of completions.

We show that (i) every completion of M is complete and M is dense in it, (ii) completions are unique up to isometry, and (iii) completions exist. We begin with the proof of (i).

Proposition 21 *Let $C = \langle C, e\rangle$ be a completion of $M = \langle M, d\rangle$. Then M is dense in C and the metric space C is complete.*

Proof. The first claim is immediate from Exercise 19, and from parts 2 and 3 in Definition 20. We show that C is complete. Let $(a_n) \subset C$ be a Cauchy sequence in the space C. Since M is dense in C, by Exercise 19 we can take points $b_n \in M$ such that $e(a_n, b_n) \le n^{-1}$ for every $n \in \mathbb{N}$. We show that (b_n) is a Cauchy sequence in M. Let $\varepsilon > 0$ be given. Then $e(a_m, a_n) \le \varepsilon$ for every indices $m, n \ge n_0$. We take an index $n_1 \ge n_0$ such that $e(a_n, b_n) \le \varepsilon$ for every $n \ge n_1$. Then, using TI, we obtain for every $m, n \ge n_1$ the bound

$$d(b_m,\, b_n) = e(b_m,\, b_n) \le e(b_m,\, a_m) + e(a_m,\, a_n) + e(a_n,\, b_n) \le \varepsilon + \varepsilon + \varepsilon = 3\varepsilon$$

and see that (b_n) is Cauchy. By part 2 of Definition 20, there is a point $a \in C$ such that

$$\lim_{n\to\infty} b_n = a\,.$$

We show that $\lim a_n = a$ as well. Let $\varepsilon > 0$ be given. We take an index $n_2 \ge n_1$ such that $e(b_n, a) \le \varepsilon$ for every $n \ge n_2$. Then for every $n \ge n_2$,

$$e(a_n,\, a) \le e(a_n,\, b_n) + e(b_n,\, a) \le \varepsilon + \varepsilon = 2\varepsilon\,.$$

We see that $\lim a_n = a$. □

To prove claim (ii), we need a lemma.

Lemma 22 *Let $\varepsilon > 0$, let $M = \langle M, d\rangle$ be a metric space, and let $a, b, c \in M$ be such that $d(b, c) \le \varepsilon$. Then we have the bound*

$$|d(a,\, b) - d(a,\, c)| \le \varepsilon\,.$$

Proof. Indeed, by TI we have inequalities $d(a, b) \le d(a, c) + d(b, c)$ and $d(a, c) \le d(a, b) + d(b, c)$. They are equivalent to the inequalities

$$d(a,\, b) - d(a,\, c) \le d(b,\, c) \le \varepsilon \ \text{ and } \ d(a,\, b) - d(a,\, c) \ge -d(b,\, c) \ge -\varepsilon\,,$$

respectively, and these are equivalent to the stated bound. □

Proposition 23 *Let $C = \langle C, e\rangle$ and $C' = \langle C', e'\rangle$ be two completions of $M = \langle M, d\rangle$. Then the metric spaces C and C' are isometric.*

Proof. We define a map $F\colon C \to C'$ and show that it is an isometry of C and C'. Let $a \in C$ be arbitrary. Since M is dense in C (Proposition 21), $a = \lim b_n$ for a sequence $(b_n) \subset M$. By Exercise 32 in lecture 1, (b_n) is a Cauchy sequence. By part 2 of Definition 20, we have $\lim b_n = a'$ for some $a' \in C'$. We define $F\colon C \to C'$ by setting

$$F(a) := a'\,.$$

We show that F does not depend on the choice of the sequence (b_n). Let $(b_n') \subset M$ be another sequence such that $\lim b_n' = a$. By TI,

$$e'(b_n,\, b_n') = d(b_n,\, b_n') = e(b_n,\, b_n') \le e(b_n,\, a) + e(a,\, b_n') \to 0 \ \text{ as } n \to \infty\,.$$

By Exercise 24, it follows that in the space C' we have $\lim b_n = \lim b_n'$.

We show that F is a bijection and that it preserves distances. To prove the injectivity of F, we take $a_0, a_1 \in C$, $a_0 \ne a_1$, and sequences $(b_n^i) \subset M$ such that $\lim b_n^i = a_i$ for $i = 0, 1$. Then there is $\varepsilon > 0$ such that $d(b_n^0, b_n^1) \ge \varepsilon$ for every large n. It follows from this that $F(a_0) \ne F(a_1)$. To prove the surjectivity of F, we need that $F(a)$ does not depend on the sequence $(b_n) \subset M$ chosen such that $\lim b_n = a$ (Exercise 25). We take any point $a' \in C'$. Since M is dense in C', we have $a' = \lim b_n'$ for a sequence $(b_n') \subset M$. Since (b_n') is Cauchy, by part 2 of Definition 20 we have $\lim b_n' = a$ for some $a \in C$. Hence $F(a) = a'$.

Finally, to prove that F preserves distances, we take two points $a_0, a_1 \in C$, and sequences $(b^i_n) \subset M$ such that $\lim b^i_n = a_i$ for $i = 0, 1$. Then

$$\begin{aligned} e(a_0,\, a_1) &= \lim_{n\to\infty} e(b^0_n,\, b^1_n) = \lim_{n\to\infty} d(b^0_n,\, b^1_n) = \lim_{n\to\infty} e'(b^0_n,\, b^1_n) = \\ &= e'(F(a_0),\, F(a_1))\,, \end{aligned}$$

and $e(a_0, a_1) = e'(F(a_0), F(a_1))$. To prove the first equality in detail, we take $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that $e(a_i,\, b^i_n) \le \varepsilon$ for $i = 0, 1$ and $n \ge n_0$. Then for every $n \ge n_0$ we indeed have, by Lemma 22 and TI in E_1, that

$$\begin{aligned} |e(a_0,\, a_1) - e(b^0_n,\, b^1_n)| &\le |e(a_0,\, a_1) - e(b^0_n,\, a_1)| + |e(b^0_n,\, a_1) - e(b^0_n,\, b^1_n)| \\ &\le \varepsilon + \varepsilon = 2\varepsilon \end{aligned}$$

(consider triangles a_0, b^0_n, a_1 and b^0_n, b^1_n, a_1). The equality $\lim e'(b^0_n, b^1_n) = e'(F(a_0), F(a_1))$ is proven in the same way, we only work in the space C' instead of C. $\square$

Exercise 24 *Prove in detail that* $\lim b_n = \lim b'_n$ *(in* C'*).*

Exercise 25 *One might think that it is possible to omit the proof that the value* $F(a) = a'$ *does not depend on the choice of the sequence* $S_a = (b_n) \subset M$ *with* $\lim b_n = a$*. We simply fix one sequence* S_a *for every* $a \in C$*. Show that this approach does not work in the proof of surjectivity of* F*.*

Finally, we prove claim (iii) that every metric space has a completion. Without this existence result, Propositions 21 and 23 could be void statements. Two sequences (a_n) and (a'_n) in a metric space $M = \langle M, d\rangle$ are *equivalent*, written $(a_n) \sim (a'_n)$, if $\lim d(a_n,\, a'_n) = 0$.

Exercise 26 *The relation* $\sim$ *is an equivalence relation.*

We denote by $\mathcal{C}(M)$ the set of all Cauchy sequences in M. We say that an *(equivalence) block* $B \in \mathcal{C}(M)/\!\sim$ is *bad* if there is no point $b \in M$ such that $\lim a_n = b$ for some (equivalently, any) sequence $(a_n) \in B$.

Theorem 27 (existence of completions) *Any metric space* $M = \langle M, d\rangle$ *has a completion* $C = \langle C, e\rangle$*.*

Proof. Let $M = \langle M, d\rangle$ be a metric space and let

$$M' := \{B \in \mathcal{C}(M)/\!\sim\colon\ B \text{ is bad}\}\,.$$

Using the device of disjoint union, recalled in Exercise 28, we may assume that $M \cap M' = \emptyset$. We set

$$C := M \cup M'\,;$$

it is a disjoint union. We first define a map $e\colon C \times C \to \mathbb{R}$. Then we show that $\langle C, e\rangle$ is a metric space. Finally, we show that the pair of $\langle M, d\rangle$ and $\langle C, e\rangle$ has properties 1–3 in Definition 20.

Let $a, b \in C$. If $a, b \in M$, we define $e(a, b) := d(a, b)$. If a is in M and $b = B \in M'$, we take any sequence $(b_n) \in B$ and set

$$e(a, B) := \lim_{n\to\infty} d(a,\, b_n) \ \ (\in \mathbb{R})\,.$$

This limit exists by Theorem 16, because the sequence $(d(a, b_n)) \subset \mathbb{R}$ is Cauchy by Lemma 22. Moreover, it is easy to show that the limit does not depend on the choice of the sequence (b_n) in B. We similarly define values $e(A, b)$ for $A \in M'$ and $b \in M$. For $A, B \in M'$ and any sequences $(a_n) \in A$ and $(b_n) \in B$ we set

$$e(A, B) := \lim_{n\to\infty} d(a_n,\, b_n) \ \ (\in \mathbb{R})\,.$$

The existence of this limit and its independence on the choice of sequences (a_n) and (b_n) is justified as before.

We show that $\langle C, e\rangle$ is a metric space. Clearly, e is a nonnegative and symmetric function because d has these properties. Let $a, b \in C$. Let $a = b$. If $a = b \in M$ then $e(a, b) = d(a, b) = 0$ because d is a metric. If $a = A = B = b \in M'$, then $e(a, b) = 0$ because we can take twice the same sequence $(a_n) = (b_n) \in A = B$ to compute the limit $e(a, b) = \lim d(a_n, b_n) = \lim d(a_n, a_n) = 0$. Let $e(a, b) = 0$. If $a, b \in M$, then $a = b$ because d is a metric. Let $a \in M$, $b = B \in M'$, and $(b_n) \in B$. Then $e(a, B) = \lim d(a, b_n) = 0$ is, in fact, impossible because then $\lim b_n = a$ (in M) and B is not a bad block; same if $a = A \in M'$ and $b \in M$. Let $a = A, b = B \in M'$, and let $(a_n) \in A$ and $(b_n) \in B$. Then $e(A, B) = \lim d(a_n, b_n) = 0$ means that $(a_n) \sim (b_n)$ and $A = B$. We show that TI holds for e. This easily follows from three facts: the definition of e, TI holds for d, and the fact that non-strict real inequalities are preserved in limits (if $x_n \le y_n$ for every n, then $\lim x_n \le \lim y_n$, if these limits exist). Thus $\langle C, e\rangle$ is a metric space.

We show that $C = \langle C, e\rangle$ is a completion of M by Definition 20. By the definition of e, the space M is a subspace of C. Let $(a_n) \in \mathcal{C}(M)$. If $\lim a_n = b$ (in M) for some $b \in M$, we are done. Else $A = [(a_n)]_\sim \in M'$. Then

$$\lim_{n\to\infty} e(A,\, a_n) = \lim_{n\to\infty}\lim_{m\to\infty} d(a_m,\, a_n) = 0$$

because (a_n) is a Cauchy sequence in M. Hence $\lim a_n = A$ (in C). Finally, to prove property 3 in Definition 20, we take a bad block $A = [(a_n)]_\sim \in M'$ and delete it from C. But then the sequence (a_n) ($\in \mathcal{C}(M)$) does not have a limit in $C \setminus \{A\}$, because in C it has the limit A, and property 2 is lost. The proof is complete. $\square$

Exercise 28 *Recall that the* disjoint union *of two sets X and Y is the set*

$$(\{0\} \times X) \cup (\{1\} \times Y)\,.$$

Explain why we can assume in the previous proof, using this construction, that $M \cap M' = \emptyset$. What would go wrong if $M \cap M' \neq \emptyset$?

Topological spaces

Topological spaces, or simply topologies, are abstractions of metric spaces to hypergraphs (systems of subsets of a given set) that enjoy all the properties of open sets. Topologies were introduced by the German mathematician *Felix Hausdorff (1868–1942)*. Recall that for any set X,

$$\mathcal{P}(x) := \{Y:\ Y \subset X\}$$

is the *power set* of X, the set of all subsets of X.

Definition 29 (topologies) *A pair $T = \langle T, \tau\rangle$ is a* topology *if $T \neq \emptyset$ is a set, $\tau \subset \mathcal{P}(T)$ is a hypergraph with the base set T, and three conditions hold.*

1. *$\emptyset, T \in \tau$.*

2. *If $A, B \in \tau$, then $A \cap B \in \tau$.*

3. *If $A_i \in \tau$ for every $i \in I$, then $\bigcup_{i\in I} A_i \in \tau$.*

The elements of τ are so called *open sets* of the topology. Their complements $T \setminus A$, $A \in \tau$, are so called *closed sets* of the topology. By Proposition 15 in lecture 1, for every metric space $M = \langle M, d\rangle$, the pair

$$\langle M, \text{open sets in } M\rangle$$

is a topology. We call it a *metric topology.* By the definitions, closed sets in metric spaces and in metric topologies coincide. Other examples of topologies are the *discrete topology* $\langle T, \mathcal{P}(T)\rangle$ and the *indiscrete topology* $\langle T, \{\emptyset, T\}\rangle$.

In metric topologies, points are closed; you show it in the next two exercises.

Exercise 30 *Every point $\{b\}$, $b \in M$, in the metric topology is closed. Prove it from the definition of closed sets as complements of open sets.*

Exercise 31 *Prove the same by means of the characterization of closed sets in metric spaces by limits of sequences (Proposition 28 in lecture 1).*

Exercise 32 *Give an example of a topology $\langle T, \tau\rangle$ such that its points $\{b\}$, $b \in T$, are not closed sets.*

In the next theorem, we identify the property that sets metric topologies apart from general topologies.

Theorem 33 (normality of metric topologies) *$M = \langle M, d\rangle$ be a metric space. The metric topology on M is <u>normal</u>: for any two disjoint closed sets $U, V \subset M$ there exist two disjoint open sets $A, B \subset M$ such that*

$$U \subset A \text{ and } V \subset B\,.$$

For the proof, we need the following lemma.

Lemma 34 *Suppose that $B_1 := B(b_1, r)$ and $B_2 := B(b_2, s)$ are two balls in a metric space $\langle M, d\rangle$ such that*

$$b_2 \notin B_1 \text{ and } b_1 \notin B_2\,.$$

Then $B(b_1, r/2) \cap B(b_2, s/2) = \emptyset$.

Proof. We have $b_1 \neq b_2$ and $t := d(b_1, b_2) > 0$. Clearly, $0 < r, s \leq t$. Suppose for the contrary that $b \in B(b_1,\, r/2) \cap B(b_2,\, s/2)$. Then TI produces the contradiction

$$t = d(b_1,\, b_2) \leq d(b_1,\, b) + d(b,\, b_2) < r/2 + s/2 \leq t/2 + t/2 = t\,.$$

□

Proof of Theorem 33. Since $U \subset M \setminus V$ and the complement is open, we can take for every $u \in U$ a ball B_u with the center u such that $B_u \cap V = \emptyset$. Similarly, for every point $v \in V$ we have a ball B_v with the center v such that $B_v \cap U = \emptyset$. Let B'_u and B'_v be these balls with their radii decreased to half. By the previous lemma, $B'_u \cap B'_v = \emptyset$ for every $u \in U$ and every $v \in V$. It follows (by Exercise 35) that

$$A := \textstyle\bigcup_{u \in U} B'_u \text{ and } B := \bigcup_{v \in V} B'_v$$

are the desired open sets. □

Exercise 35 *Show that the sets A and B are open, have empty intersection, $U \subset A$, and $V \subset B$.*

A topology $\langle T, \tau \rangle$ is a *Hausdorff topology*, also called a *T_2-space*, if for every two points $a, b \in T$, $a \neq b$, there exist open sets $A, B \in \tau$ such that

$$a \in A,\;\; b \in B,\;\; \text{and}\;\; A \cap B = \emptyset\,.$$

By Exercises 30 and 31, and by Theorem 33, we see that every metric topology is Hausdorff. This property of metric topology is crucial for proving the uniqueness of limits of sequences.

Exercise 36 *Give a simple direct proof, not using Theorem 33, that metric topologies are Hausdorff.*

A topology $T = \langle T, \tau \rangle$ is *normal*, also called a *T_4-space*, if for every two disjoint closed sets U and V in T there exist open sets $A, B \in \tau$ such that

$$U \subset A,\;\; V \subset B,\;\; \text{and}\;\; A \cap B = \emptyset\,.$$

By Theorem 33, every metric topology is normal.

Exercise 37 *Consider a hypergraph $T = \langle \mathbb{N}, \tau \rangle$ such that $A \subset \mathbb{N}$ is in τ iff $A = \emptyset$ or $A = \mathbb{N} \setminus X$ for a finite set $X \subset \mathbb{N}$. Show that T is a non-Hausdorff and non-normal topology.*

Normed vector spaces and Hilbert spaces

Topologies are more general structures than metric spaces. Now we review some structures that are more special than metric spaces, with $\mathbb{R}^n$ being an example. We assume that the reader is familiar with vector spaces over fields; in our case the relevant field is $\mathbb{R}$. We briefly recall them. A *vector space V over the field F* is a quadruple

$$V = \langle V,\, 0_V,\, +,\, \{\lambda_r\colon\ r \in F\}\rangle$$

of the base set $V \neq \emptyset$ of *vectors*, the *zero vector* $0_V \in V$, the operation $+\colon V \times V \to V$ of *addition*, and the set of unary operations $\lambda_r\colon V \to V$. We call λ_r the *multiplication by scalar r*, and we write the values $\lambda_r(v)$, $v \in V$ and $r \in F$, briefly as rv ($\in V$). We assume that the reader is familiar with the axioms/properties of vector spaces. There are many examples of vector spaces, but the one important for us is $V = \mathbb{R}^n$, with $0_V = \langle 0, \dots, 0\rangle$, coordinate-wise addition, and with coordinate-wise multiplication by scalars in $\mathbb{R}$.

Definition 38 (normed vector spaces) *A* norm *on a vector space V over the field $\mathbb{R}$ is a map $\|\cdot\|\colon V \to \mathbb{R}$ with the following three properties ($u, v \in V$ and $a \in \mathbb{R}$).*

1. *Always $\|u\| \ge 0$ and $\|u\| = 0$ iff $u = 0_V$.*
2. *Always $\|au\| = |a| \cdot \|u\|$.*
3. *Always $\|u + v\| \le \|u\| + \|v\|$.*

We call the pair $\langle V, \|\cdot\|\rangle$ a normed vector space*.*

The third property is another incarnation of TI. For any normed vector space $\langle V, \|\cdot\|\rangle$ we define a function $d_V\colon V \times V \to \mathbb{R}$ by

$$d_V(u,\, v) := \|u - v\|\,.$$

Exercise 39 *What is the operation $-$ appearing in the definition of d_V? How does it relate to addition?*

Normed vector spaces are related to metric spaces in the next exercise.

Exercise 40 *Show that for every normed vector space $\langle V, \|\cdot\|\rangle$, the pair*

$$\langle V,\, d_V(u,\, v)\rangle = \langle V,\, \|u - v\|\rangle$$

is a metric space.

If the metric space $\langle V, d_V\rangle$ is complete, we say that the normed vector space $\langle V, \|\cdot\|\rangle$ is a *Banach space*.

The reader is probably familiar with the notion of *inner product* (also called *scalar product*) on a vector space V. It is a function

$$\mathrm{sp}\colon V \times V \to \mathbb{R}$$

that satisfies three conditions (we use the notation $\mathrm{sp}(u, v)$ instead of the more common $\langle u, v\rangle$ to avoid conflict with the notation for tuples). The properties of a scalar product are as follows.

1. sp is a bilinear map: $\mathrm{sp}(u, av + bv') = a\mathrm{sp}(u, v) + b\mathrm{sp}(u, v')$ for every $u, v, v' \in V$ and $a, b \in \mathbb{R}$. The same holds in the first variable u.
2. sp is symmetric: $\mathrm{sp}(u, v) = \mathrm{sp}(v, u)$ for every $u, v \in V$.
3. sp is positively definite: $\mathrm{sp}(u, u) \geq 0$ for every $u \in V$, with equality only for $u = 0_V$.

For example, the vector space $\mathbb{R}^n$ has the scalar product

$$\mathrm{sp}(\overline{u},\, \overline{v}) := \textstyle\sum_{i=1}^{n} u_i v_i\,.$$

Exercise 41 *Show that it is a scalar product.*

Unlike metrics and norms, scalar products may have negative values. As we promised in lecture 1, we generalize the Cauchy–Schwarz inequality to vector spaces with a scalar product.

Theorem 42 (CSI for scalar products) *Let $\mathrm{sp}(u, v)$ be a scalar product on a vector space V. Then*

$$\mathrm{sp}(u, v) \leq \sqrt{\mathrm{sp}(u, u)\cdot \mathrm{sp}(v, v)}\ \ \big(\iff |\mathrm{sp}(u, v)| \leq \sqrt{\mathrm{sp}(u, u)\cdot \mathrm{sp}(v, v)}\,\big)$$

for every vectors $u, v \in V$. Moreover, the inequality becomes an equality if and only if the vectors u and v are linearly dependent, i.e., one is a scalar multiple of another.

Proof. Let $u, v \in V$ be any vectors. If $v = 0_V$, the CS inequality holds trivially as the equality $0 = 0$. Let $v \neq 0_V$ and $\alpha := \mathrm{sp}(u, v)/\mathrm{sp}(v, v)$ $(\in \mathbb{R})$. We have

$$\begin{aligned} 0 &\le \mathrm{sp}(u - \alpha v,\, u - \alpha v) \\ &= \mathrm{sp}(u,\, u) - 2\alpha \mathrm{sp}(u,\, v) + \alpha^2 \mathrm{sp}(v,\, v) \\ &= \mathrm{sp}(u,\, u) - \mathrm{sp}(u,\, v)^2/\mathrm{sp}(v,\, v)\,. \end{aligned}$$

The first inequality follows from the positive definiteness of $\mathrm{sp}(\cdot, \cdot)$, the second equality follows from the bilinearity and symmetry of $\mathrm{sp}(\cdot, \cdot)$, and the last, third equality is due to the choice of the scalar α. The obtained inequality

$$0 \le \mathrm{sp}(u,\, u) - \mathrm{sp}(u,\, v)^2/\mathrm{sp}(v,\, v)$$

is equivalent to

$$\mathrm{sp}(u,\, v)^2 \le \mathrm{sp}(u,\, u) \cdot \mathrm{sp}(v,\, v) \iff |\mathrm{sp}(u, v)| \le \sqrt{\mathrm{sp}(u, u) \cdot \mathrm{sp}(v, v)}\,.$$

In view of Exercise 43, this is what we wanted to prove. The last claim is clear: the inequality holds as an equality iff $u - \alpha v = 0_V$. $\square$

Exercise 43 *Show that the two versions of* CSI *for scalar products, with and without the absolute value, are equivalent.*

For any vector space V with a scalar product $\mathrm{sp}(u, v)$ we define a function $\|\cdot\|_V \colon V \to \mathbb{R}$ by

$$\|u\|_V := \sqrt{\mathrm{sp}(u,\, u)}\,.$$

Vector spaces with a scalar product are related to normed vector spaces and hence to metric spaces in the next exercise.

Exercise 44 *Show that for every vector space V with a scalar product, the pair $\langle V, \|\cdot\|_V\rangle$ is a normed vector space.*

It follows that if V is a vector space with a scalar product $\mathrm{sp}(u, v)$, then the pair

$$\langle V,\, d_V(u,\, v)\rangle := \left\langle V,\, \sqrt{\mathrm{sp}(u - v,\, u - v)}\right\rangle$$

is a metric space.

Now we recall Hilbert spaces. They are named after *David Hilbert (1862–1943)*, who was a highly versatile, creative, and influential mathematician of his time.

Definition 45 (Hilbert spaces) *Let V be a vector space with a scalar product* $\mathrm{sp}(u, v)$*. If the metric space*

$$\langle V,\, d_V(u,\, v)\rangle = \left\langle V,\, \sqrt{\mathrm{sp}(u-v, u-v)}\right\rangle$$

is complete, we call the pair $\langle V, \mathrm{sp}(u, v)\rangle$ Hilbert space*.*

To sum up, we have described a hierarchy of mathematical structures:

$$\begin{aligned}&\mathbb{R}^n \subset \text{Hilbert spaces} \subset \text{normed vector spaces} \subset \\ &\subset \text{metric spaces} \subset \text{topologies}\,.\end{aligned}$$

Compact sets and compact spaces

We introduce the second important class of metric spaces: compact spaces. Recall that a sequence (b_n) is a *subsequence* of a sequence (a_n), if there exists a sequence of indices $m_1 < m_2 < \ldots$ such that

$$b_n = a_{m_n} \text{ for every } n \in \mathbb{N}\,.$$

Definition 46 (compactness) *A set $X \subset M$ in a metric space $\langle M, d\rangle$ is compact if every sequence $(a_n) \subset X$ has a subsequence (a_{m_n}) that has a limit*

$$\lim_{n\to\infty} a_{m_n} \in X\,.$$

If the set M is compact, we say that the space M is compact.

Exercise 47 *Prove that a discrete metric space $\langle M, d\rangle$ is compact if and only if the base set M is finite.*