Mathematical Analysis 2 (MA2) – Lecture 1

Martin Klazar

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Complete texts of the lectures are on `https://kam.mff.cuni.cz/~klazar/MAII26en.html` and students are supposed to study them carefully; the slides are just outlines.

This lecture

The course MA2

In this follow-up of the course MA1, we study functions of the form

$$f\colon X \to \mathbb{R} \ \text{ where } \ X \subset \mathbb{R}^n\,.$$

The course MA2 has three parts:

- Metric spaces: complete, compact, connected ($\mathbb{R}^n$ is a metric space) – ca. 3 lectures
- Multivariate differential calculus – ca. 5 lectures
- Multivariate integral calculus – ca. 5 lectures

Exam: see `https://kam.mff.cuni.cz/~klazar/MAII26en.html`

Part 1. Metric spaces

Metric spaces belong to the most common and useful structures in mathematics and computer science. They were introduced by the French mathematician *Maurice R. Fréchet (1878–1973)*.

Definition (metric spaces)

A *metric space* is a pair $\langle M, d\rangle$ where $M \neq \emptyset$ and function $d\colon M \times M \to \mathbb{R}$ (called a *metric* or *distance*) satisfies, for all $x, y, z \in M$:

1. $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$;
2. (symmetry) $d(x, y) = d(y, x)$ and
3. (triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$.

In MA1, we already worked with the metric space $\langle \mathbb{R}, |x - y|\rangle$. More examples: the *discrete spaces* $\langle M, d\rangle$ where $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$.

Every definition of a class of mathematical structures should be followed by the definition of their isomorphisms. For metric spaces, these are so called isometries.

Definition (Isometries)

Two metric spaces $\langle M, d\rangle$ and $\langle N, e\rangle$ are *isometric* if there is a bijection $F\colon M \to N$ such that

$$d(x, y) = e(F(x), F(y)) \quad \text{for all } x, y \in M\,.$$

We say that F is an *isometry* of the spaces M and N.

Note that F^{-1} is an isometry too. An example: the spaces $E_m \times E_n$ and E_{m+n} are isometric; we define the product of spaces $\times$ and the Euclidean spaces E_n later.

Euclidean metric e_n on $\mathbb{R}^n$

For $n \in \mathbb{N}$ $(= \{1, 2, \dots\})$, the *Euclidean (metric) space* E_n is the pair $\langle \mathbb{R}^n, e_n \rangle$, where we define $e_n \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$e_n(\langle x_1, \dots, x_n \rangle, \langle y_1, \dots, y_n \rangle) := \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}\,.$$

For $n = 1$, $e_1(x, y) = |x - y|$ is the familiar metric on the real line. For $n = 2$,

$$e_2(\overline{x}, \overline{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$$

is the familiar Euclidean distance in the plane. It is easy to see that $e_n(\overline{x}, \overline{y}) \ge 0$, with equality only for $\overline{x} = \overline{y}$, and that $e_n(\overline{x}, \overline{y}) = e_n(\overline{y}, \overline{x})$. We prove that the triangle inequality holds for e_n.

Cauchy–Schwarz inequality (CSI)

We start from another inequality, the *Cauchy–Schwarz inequality* that is attributed to the French mathematician *Augustin-Louis Cauchy (1789–1857)* (he lived for several years in Prague) and the German mathematician *Hermann Schwarz (1843–1921)*.

Theorem (CSI)

For any real numbers x_i, y_i $(i = 1, \ldots, n)$:

$$\sum_{i=1}^{n} x_i y_i \le \sqrt{\sum_{i=1}^{n} x_i^2 \cdot \sum_{i=1}^{n} y_i^2}\,.$$

Proof ideas: transform it to the equivalent form $\sum_j z_j^2 \ge 0$, and then use that $z^2 \ge 0$ for every $z \in \mathbb{R}$. CSI generalizes to vector spaces with a scalar product.

Triangle inequality for e_n

We deduce the triangle inequality for the function e_n from the Cauchy–Schwarz inequality.

Corollary

For any three n-tuples $\overline{x}, \overline{y}, \overline{z} \in \mathbb{R}^n$ *we have*

$$e_n(\overline{x}, \overline{y}) \le e_n(\overline{x}, \overline{z}) + e_n(\overline{z}, \overline{y})\,.$$

Proof ideas: we rewrite it in terms of variables $u_i = x_i - z_i$ and $v_i = z_i - y_i$ as $\sqrt{\sum_{i=1}^n (u_i + v_i)^2} \le \sqrt{\sum_{i=1}^n u_i^2} + \sqrt{\sum_{i=1}^n u_i^2}$, square both sides, and apply CSI.

Thus $\langle \mathbb{R}^n, e_n \rangle$ is a metric space. We denote it by E_n and call it the n-dimensional *Euclidean space*. More generally, we call any subspace (to be defined) of E_n a *Euclidean space*.

Other metrics on $\mathbb{R}^n$

One can define many other metrics on $\mathbb{R}^n$. For example:

- Maximum metric:
$$e_{n,\,\max}(\overline{x},\overline{y}) := \max_{1\le i\le n} |x_i - y_i|\,.$$
- Sum metric:
$$e_{n,\,\mathrm{sum}}(\overline{x},\overline{y}) := \sum_{i=1}^{n} |x_i - y_i|\,.$$
- L_p metrics ($p \ge 1$ is real):
$$e_{n,\,p}(\overline{x},\overline{y}) := \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}\,.$$

Proofs that the first two are metrics are left to you as exercises. The proof (of TI) for the L_p metrics is more involved. Note that $e_n = e_{n,2}$.

Subspaces

We introduce two operations by which one can produce new metric spaces from old ones. The first operation is that of a subspace.

If $\langle M, d\rangle$ is a metric space and $\emptyset \neq A \subset M$, define the *subspace of M (induced by A)* by

$$M \,|\, A := \langle A, d'\rangle \ \text{ where } \ d'\colon A \times A \to \mathbb{R} \ \text{ with } \ d'(x, y) := d(x, y)\,.$$

So we just restrict the distance d to $A \times A$. It is an easy exercise to show that $M \,|\, A$ is a metric space.

As we already mentioned, we call every subspace of E_n a *Euclidean (metric) space*.

Products

The second operation on metric spaces is the product of two spaces. For metric spaces $\langle M, d\rangle$ and $\langle N, e\rangle$ we define the function $d'\colon M \times N \to \mathbb{R}$ by

$$d'(\langle a, b\rangle, \langle a', b'\rangle) := \sqrt{d(a, a')^2 + e(b, b')^2}\,.$$

We call the metric space $\langle M \times N, d'\rangle$ the *product of M and N*.

Proposition

The product of two metric spaces satisfies TI *and is a metric space.*

Proof ideas: triangle inequality in E_2; d and e are metrics.

Proposition

$E_m \times E_n$ is isometric to E_{m+n}.

Proof ideas: the isometry is given by the concatenation of coordinates.

Balls, open and closed sets

We introduce two important families of subsets in any metric space, the open and closed sets. Let $\langle M, d\rangle$ be a metric space, $b \in M$, and let $r > 0$ be a real number. The set

$$B_M(b,\ r) = B(b,\ r) := \{a \in M\colon d(a,\ b) < r\}$$

is the *(open) ball* with the *center* b and *radius* r. Note that $b \in B(b, r)$. *Closed balls* are the sets $\overline{B}_M(b, r) = \overline{B}(b, r) := \{a \in M\colon d(a, b) \le r\}$.

Definition (open and closed sets)

A set $X \subset M$ is *open* if $\forall b \in X\, \exists r > 0\colon B(b, r) \subset X$.
A set $X \subset M$ is *closed* if $M \setminus X$ is open.

As an exercise, you can show that every ball is an open set and that every closed ball is a closed set.

Properties of open and closed sets

Proposition

Let $\langle M, d\rangle$ be a metric space. Open sets in M have the following properties.

- *$\emptyset$ and M are open.*
- *Finite intersections of open sets are open.*
- *Arbitrary unions of open sets are open. We define the empty union as $\emptyset$.*

Proposition

Let $\langle M, d\rangle$ be a metric space. Closed sets in M have the following properties.

- *$\emptyset$ and M are closed.*
- *Finite unions of closed sets are closed.*
- *Arbitrary intersections of closed sets are closed. We define the empty intersection as M.*

Proof ideas: the definitions and de Morgan identities ($M \setminus \bigcup_j A_j = \bigcap_j (M \setminus A_j)$).

Continuous maps between metric spaces

We give the classical ε-δ definition of continuity.

Definition (continuity at a point)

Let $\langle M, d\rangle$ and $\langle N, e\rangle$ be two metric spaces, $b \in X \subset M$, and let $f\colon X \to N$. We say that the map f is *continuous at the point* b if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$f[B_M(b,\,\delta)] \subset B_N(f(b),\,\varepsilon)\,.$$

Our images of sets by maps generalize the standard images: if $g\colon A \to B$ is a function and C is *any* set, then $g[C] := \{g(x)\colon x \in A \cap C\}$ $(\subset B)$. Standardly it is (uselessly) required that $C \subset A$. This is inconvenient here because in general $B_M(b,\delta) \not\subset X$. Since f can be viewed as going from the subspace $M\,|\,X$, we can restrict ourselves to everywhere defined maps.

Definition (continuous maps)

A map $f\colon M \to N$ between metric spaces is *continuous* if it is continuous at every point of M.

Continuity via preimages of open sets and via Heine's definition

We give another two equivalent formulations for continuity.

Proposition

A map $f\colon M \to N$ between metric spaces is continuous if and only if $f^{-1}[B]$ is an open set in M for every open set B in N.

Proof ideas: balls are open sets, and open sets are defined via balls. Show as an exercise that the equivalence also holds with closed sets. If $g\colon A \to B$ and C is any set, then the *preimage of C by g* is $g^{-1}[C] := \{x \in A\colon g(x) \in C\}$ $(\subset A)$.

Theorem (Heine's definition)

A map $f\colon M \to N$ between metric spaces is continuous at a point $b \in M$ if and only if $\lim f(a_n) = f(b)$ for every sequence $(a_n) \subset M$ such that $\lim a_n = b$.

Proof ideas: like in MA1. The theorem relates to the German mathematician *Eduard Heine (1821–1881)*. We define limits of sequence in metric spaces next.

Limits of sequences and Cauchy sequences

Let $M = \langle M, d\rangle$ be a metric space, $b \in M$, and let $(a_n) \subset M$ be a sequence of points in it. If for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for every index $n \ge n_0$ we have

$$d(a_n,\ b) \le \varepsilon\,,$$

we write $\lim_{n\to\infty} a_n = b$ or $\lim a_n = b$ or $a_n \to b$ and say that *the sequence* (a_n) *has limit* b. You can show as an exercise that

$$\lim_{n\to\infty} a_n = b \iff \lim_{n\to\infty} d(a_n,\ b) = 0$$

(these are two different limits), and that limits are uniquely determined. We say that the sequence (a_n) is *Cauchy* (i.e., is a Cauchy sequence) if for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for every two indices $m, n \ge n_0$ we have

$$d(a_m,\ a_n) \le \varepsilon\,.$$

Closed sets via limits

We prove the following proposition.

Proposition

A set $X \subset M$ in a metric space is closed $\iff$ whenever a sequence $(a_n) \subset X$ has a limit, the limit lies in X.

Proof.

Suppose that X is closed, $(a_n) \subset X$ has a limit $\lim a_n = b$, but $b \in M \setminus X$. Since the last set is open, we have $B(b, r) \subset M \setminus X$ for some $r > 0$. By the definition of limit, $a_n \in B(b, r)$ and hence $a_n \notin X$ for every large n, which is a contradiction. We see that $b \in X$.
Suppose that X is not closed. Then the complement $M \setminus X$ is not open, which means that there exists a point $b \in M \setminus X$ such that for every $n \in \mathbb{N}$ we have $B(b, 1/n) \cap X \neq \emptyset$. Using the *axiom of choice*, we select for every n a point $a_n \in B(b, 1/n) \cap X$. In this way we get a sequence (a_n) such that $(a_n) \subset X$ and $\lim a_n = b$. However, $b \notin X$. □

Completeness

We define the first of three important classes of metric spaces, the complete spaces. The other two classes are compact spaces and connected spaces.

Definition (completeness)

A set $X \subset M$ in a metric space is *complete* if every Cauchy sequence in X has a limit in X. The space M is *complete* if M is complete as a set.

Example: $\langle \mathbb{R}, |x - y| \rangle$ is a complete metric space. Show as an exercise that discrete spaces (defined at the beginning) are complete.

Proposition

If M is a complete metric space and $X \subset M$, then X is complete $\iff$ X is closed.

Proof ideas: this follows easily from the previous proposition.

Wrap-up

Today:

- Metric spaces and isometries
- The triangle inequality for Euclidean metric via CSI
- Subspaces and products
- Balls, open and closed sets
- Continuous maps
- Limits of sequences and Cauchy sequences
- Complete sets and spaces