MA2 − lecture 1

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Abstract

We define metric spaces and isometries between them. We introduce subspaces and products of metric spaces. We prove the triangle inequality for the Euclidean metric on $\mathbb{R}^n$. We introduce balls, open sets, and closed sets. We characterize continuous maps by preimages of open sets and in the Heine way. We define complete sets in metric spaces, and prove that in a complete space a set is complete if and only if it is closed.

In this course *Mathematical Analysis 2*, we extend both differential and integral calculus in one variable, which are topics of the course *Mathematical Analysis 1*, to several variables. Main objects in *MA2* will be functions of the type

$$f\colon M \to \mathbb{R} \ \text{ with } \ M \subset \mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies of } \mathbb{R}} \ \text{ and } \ n \in \mathbb{N}\ (= \{1,\, 2,\, \dots\})\,.$$

We explain the used notation. Recall that a *function* or a *map* $g\colon X \to Y$, where X and Y are sets, is any set

$$g \subset X \times Y \quad (= \{\langle x,\, y\rangle\colon\ x \in X,\, y \in Y\})$$

such that for every element $x \in X$ there is a *unique* element $y \in Y$, denoted by $g(x)$, for which the pair $\langle x, y\rangle \in g$. By $\mathbb{R}$ we denote the *complete ordered field of real numbers.* Its completeness means that every nonempty and upper-bounded set of real numbers has in the standard linear order $\langle \mathbb{R}, <\rangle$ a supremum. It follows from this that every Cauchy sequence $(a_n) \subset \mathbb{R}$ has a (finite) limit $\lim_{n\to\infty} a_n \in \mathbb{R}$. By $\mathbb{N}$ we denote the set of natural numbers.

In order to work with the functions f, we first have to investigate the structure and properties of the set/space $\mathbb{R}^n$. Soon we see that $\mathbb{R}^n$ is in a natural way a metric space; later we mention more classes of structures which are exemplified by $\mathbb{R}^n$. This course will have three parts.

- Part 1 (ca. 3 lectures) Metric spaces: complete, compact, and connected. We also mention definitions of topological spaces, normed vector spaces, and Hilbert spaces. $\mathbb{R}^n$ is an example of any of these.
- Part 2 (ca. 5 lectures) Multivariate differential calculus
- Part 3 (ca. 5 lectures) Multivariate integral calculus

On Exercises. Many exercises will be interspersed in the texts of lectures. Some of them will be solved in the tutorials, and/or the tutors can use them for homework. By solving them, the reader will gain a better understanding of the presented topics.

Part 1. Metric spaces

We begin the course by an introduction to the theory of metric spaces. These useful mathematical structures, with many applications in computer science, were introduced by the French mathematician *Maurice R. Fréchet (1878–1973).*

Metric spaces, especially Euclidean ones

In a *metric space* $M = \langle M, d\rangle$, the[1] first component M is a nonempty set and $d\colon M \times M \to \mathbb{R}$ is a function, called a *distance* or a *metric*. It is required that d has three properties ($x, y, z \in M$).

1. Always $d(x, y) \geq 0$. We have $d(x, y) = 0$ iff $x = y$.
2. (symmetry) Always $d(x, y) = d(y, x)$.
3. (the triangle inequality, or TI) Always $d(x, y) \leq d(x, z) + d(z, y)$.

In *MA1*, we worked with the metric space $\langle \mathbb{R}, |x - y|\rangle$. Examples of metric spaces are also the *discrete spaces* $\langle M, d\rangle$, in which $d(x, y) = 1$ for $x \neq y$ and $d(x, x) = 0$.

[1] Notation $M = \langle M, d\rangle$ means that we use the symbol M to denote both the whole pair, i.e. the metric space, and the base set.

Exercise 1 *The nonnegativity $d(x,y) \ge 0$ follows from other properties of metric.*

Exercise 2 *Prove the following generalization of* TI. *Let $x_1, \ldots, x_n$, $n \ge 2$, be points in a metric space $\langle M, d\rangle$. Then $d(x_1, x_n) \le \sum_{i=2}^{n} d(x_{i-1}, x_i)$.*

When a family of mathematical structures is defined, another definition should follow: that of isomorphisms of the just introduced structures. We define isomorphisms of metric spaces.

Definition 3 (isometries) *Let $M = \langle M, d\rangle$ and $N = \langle N, e\rangle$ be two metric spaces. We say that they are* <u>*isometric*</u> *if there exists a bijection $F\colon M \to N$ such that for every $x, y \in M$,*

$$d(x,\, y) = e(F(x),\, F(y))\,.$$

The map F is called an <u>*isometry*</u> *of the spaces M and N.*

We present examples of isometries in Proposition 11 in this lecture and in Proposition 23 in the next lecture.

Exercise 4 *If F is an isometry of two metric spaces, then so is the inverse map F^{-1}.*

We define the *Euclidean metric* e_n on $\mathbb{R}^n$ by

$$e_n(\overline{x},\, \overline{y}) = e_n(\langle x_1,\, \ldots,\, x_n\rangle,\, \langle y_1,\, \ldots,\, y_n\rangle) := \sqrt{\textstyle\sum_{i=1}^{n}(x_i - y_i)^2}\,.$$

Thus $e_1(x,y) = |x-y|$ is the distance on $\mathbb{R}$ we know from *MA1*, and e_2 is the Euclidean distance in the plane. It is easy to see that e_n has properties 1 and 2 of a metric. It remains to prove the triangle inequality. We derive it from the famous *Cauchy–Schwarz inequality*. It is attributed to the French mathematician *Augustin-Louis Cauchy (1789–1857)* (he lived for several years in Prague) and the German mathematician *Hermann Schwarz (1843–1921)*.

Theorem 5 (CSI) *Let $x_1, y_1, \ldots, x_n, y_n$ be $2n$ real numbers ($n \in \mathbb{N}$). Then*

$$\textstyle\sum_{i=1}^{n} x_i y_i \le \sqrt{\textstyle\sum_{i=1}^{n} x_i^2 \cdot \sum_{j=1}^{n} y_j^2}\,.$$

Proof. We square both sides and see that CSI holds if the inequality

$$\sum_{i=1}^n x_i^2 y_i^2 + 2 \sum_{\substack{i,j=1 \\ i<j}}^n x_i y_i x_j y_j \le \sum_{i=1}^n x_i^2 \cdot \sum_{j=1}^n y_j^2$$

holds. This is because $z \le \sqrt{z^2}$ for every $z \in \mathbb{R}$ and $f(z) = \sqrt{z}$ increases on $\mathbb{R}_{\ge 0}$. The last displayed inequality is equivalent to the inequality

$$0 \le \sum_{\substack{i,j=1 \\ i<j}}^n (x_i y_j - x_j y_i)^2 \,.$$

This inequality holds because $z^2 \ge 0$ for every $z \in \mathbb{R}$. $\square$

In the next lecture, we generalize CSI to vector space with inner product.

Corollary 6 *For every three n-tuples $\overline{x}$, $\overline{y}$, and $\overline{z}$ in $\mathbb{R}^n$ we have*

$$e_n(\overline{x},\, \overline{y}) \le e_n(\overline{x},\, \overline{z}) + e_n(\overline{z},\, \overline{y}) \,.$$

Proof. Writing $u_i := x_i - z_i$ and $v_i := z_i - y_i$, we see that we need to prove that

$$\sqrt{\sum_{i=1}^n (u_i + v_i)^2} \le \sqrt{\sum_{i=1}^n u_i^2} + \sqrt{\sum_{j=1}^n v_j^2} \,.$$

We square both sides and see that the last inequality is equivalent to the inequality

$$\sum_{i=1}^n (u_i + v_i)^2 \le \sum_{i=1}^n u_i^2 + \sum_{j=1}^n v_j^2 + 2\sqrt{\sum_{i=1}^n u_i^2 \cdot \sum_{j=1}^n v_j^2}$$
$$\iff \sum_{i=1}^n u_i v_i \le \sqrt{\sum_{i=1}^n u_i^2 \cdot \sum_{j=1}^n v_j^2} \,.$$

The last inequality is the Cauchy–Schwarz inequality. $\square$

The pair $\langle \mathbb{R}^n, e_n \rangle$ is therefore a metric space. We call it an *(n-dimensional) Euclidean space* and denote it by E_n. We could give many more examples of metric spaces, but we save it for the course *MA3*. Here we restrict ourselves to several examples of other metrics on $\mathbb{R}^n$. We call the functions

$$e_{n,\max}(\overline{x},\, \overline{y}) := \max_{1 \le i \le n} |x_i - y_i|,\;\; e_{n,\mathrm{sum}}(\overline{x},\, \overline{y}) := \sum_{i=1}^n |x_i - y_i| \,,$$

and $e_{n,p}(\overline{x}, \overline{y}) := \left(\sum_{i=1}^n |x_i - y_i|^p\right)^{1/p}$, where $p \ge 1$ is real, the *maximum metric*, the *sum metric*, and the L_p *metric*, respectively. Note that $e_n = e_{n,2}$.

Exercise 7 *Prove that* $e_{n,\max}$ *and* $e_{n,\mathrm{sum}}$ *are metrics on* $\mathbb{R}^n$*, that is to say that they satisfy* TI.

The proof of TI for L_p distances is more involved. Later, we explain that all distances e_n, $e_{n,\max}$, $e_{n,\mathrm{sum}}$, and $e_{n,p}$ are, in a sense, very similar.

Two operations with metric spaces

Let $M = \langle M, d\rangle$ and $N = \langle N, e\rangle$ be metric spaces, and let $\emptyset \neq A \subset M$. The *subspace* of M induced by the subset A is the pair

$$M \,|\, A := \langle A,\, d'\rangle \;\text{ where }\; d'\colon A \times A \to \mathbb{R} \;\text{ is given by }\; d'(x,\, y) := d(x,\, y)$$

— we simply restrict the metric d to $A \times A$.

Exercise 8 *Show that* $M \,|\, A$ *is a metric space.*

The second operation is the product of two metric spaces. The *product* of metric spaces M and N is the pair

$$\langle M \times N,\, d'\rangle \text{ where the map } d'\colon (M \times N) \times (M \times N) \to \mathbb{R}$$

is given by

$$d'(\langle a,\, b\rangle,\, \langle a',\, b'\rangle) := \sqrt{d(a,\, a')^2 + e(b,\, b')^2}\,.$$

Proposition 9 *The product of two metric spaces is a metric space.*

Proof. It is easy to show, since d and e are metrics, that the map d' has properties 1 and 2 of distance (see Exercise 10). We prove that TI holds for d'. Indeed, for any pairs $\overline{u} = \langle a,\, b\rangle$, $\overline{v} = \langle a',\, b'\rangle$, and $\overline{w} = \langle a'',\, b''\rangle$ in $M \times N$ we have

$$\begin{aligned} d'(\overline{u},\, \overline{v}) &= \sqrt{d(a,\, a')^2 + e(b,\, b')^2} \\ &\leq \sqrt{(d(a,\, a'') + d(a'',\, a'))^2 + (e(b,\, b'') + e(b'',\, b'))^2} \\ &\leq \sqrt{d(a,\, a'')^2 + e(b,\, b'')^2} + \sqrt{d(a'',\, a')^2 + e(b'',\, b')^2} \\ &= d'(\overline{u},\, \overline{w}) + d'(\overline{w},\, \overline{v})\,. \end{aligned}$$

The first equality follows from the definition of d'. The second inequality follows from two applications of TI for the metrics d and e. The third inequality is an instance of the first inequality in the proof of Corollary 6. The last, fourth equality follows from the definition of d'. $\square$

Exercise 10 *Prove that the map d' has properties 1 and 2 of distance.*

Proposition 11 *The product $E_m \times E_n$ of Euclidean metric spaces is isometric to the Euclidean metric space E_{m+n}.*

Proof. We define a map $F\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^{m+n}$ by

$$F(\langle\langle x_1,\, \ldots,\, x_m\rangle,\, \langle y_1,\, \ldots,\, y_n\rangle\rangle) := \langle x_1,\, \ldots,\, x_m,\ y_1,\, \ldots,\, y_n\rangle\,,$$

and show that it is an isometry between the product space $E_m \times E_n$ and the Euclidean space E_{m+n}. It is clear that F is a bijection (by Exercise 12). We show that F preserves distances. For $\overline{u} := \langle x_1,\, \ldots,\, x_m\rangle$, $\overline{u}' := \langle x_1',\, \ldots,\, x_m'\rangle$, $\overline{v} := \langle y_1,\, \ldots,\, y_n\rangle$, and $\overline{v}' := \langle y_1',\, \ldots,\, y_n'\rangle$ we indeed have

$$\begin{array}{rcl} d'(\langle\overline{u},\, \overline{v}\rangle,\, \langle\overline{u}',\, \overline{v}'\rangle) & = & \sqrt{e_m(\overline{u},\, \overline{u}')^2 + e_n(\overline{v},\, \overline{v}')^2} \\ & = & \sqrt{\sum_{i=1}^{m}(x_i - x_i')^2 + \sum_{i=1}^{n}(y_i - y_i')^2} \\ & = & \sqrt{\sum_{i=1}^{m+n}(F(\langle\overline{u},\, \overline{v}\rangle)_i - F(\langle\overline{u}',\, \overline{v}'\rangle)_i)^2} \\ & = & e_{m+n}\big(F(\langle\overline{u},\, \overline{v}\rangle),\, F(\langle\overline{u}',\, \overline{v}'\rangle)\big)\,. \end{array}$$

The first equality follows from the definition of the map d'. The second equality follows from the definition of metrics e_m and e_n. In the third equality, we use the definition of F and the coordinate notation

$$F(\langle\overline{x},\, \overline{y}\rangle) = \langle F(\langle\overline{x},\, \overline{y}\rangle)_1,\, \ldots,\, F(\langle\overline{x},\, \overline{y}\rangle)_{m+n}\rangle\,.$$

The last, fourth equality follows from the definition of e_{m+n}. □

Exercise 12 *Prove that the above function F is a bijection.*

Balls, open sets, closed sets

Let $M = \langle M, d\rangle$ be a metric space, $b \in M$ be a point in it, and $r > 0$ be a real number. The set

$$B_M(b,\, r) = B(b,\, r) := \{a \in M\colon\ d(a,\, b) < r\}\ \ (\subset M)$$

is an *(open) ball* in M with the *center* b and *radius* r. Note that $b \in B(b, r)$. Also, $0 < r \le s \Rightarrow B(b, r) \subset B(b, s)$. What are *closed balls?* Subsets of M of the form

$$\overline{B}_M(b,\, r) = \overline{B}(b,\, r) := \{a \in M\colon\ d(a,\, b) \le r\}\,.$$

Definition 13 (open and closed sets) *A set $X \subset M$ in a metric space $\langle M, d\rangle$ is* <u>*open*</u> *if*

$$\forall\, b \in X\, \exists\, r > 0\colon\ B(b,\, r) \subset X\,.$$

We say that a set $X \subset M$ is <u>*closed*</u> *if its complement $M \setminus X$ is open.*

Exercise 14 *Show that every ball is an open set, and that every closed ball is a closed set.*

We obtain several properties of open and closed sets.

Proposition 15 *Let $M = \langle M, d\rangle$ be a metric space. The family of open sets in M has the following properties.*

1. *The sets $\emptyset$ and M are open.*
2. *If X and Y are open sets, then their intersection $X \cap Y$ is open.*
3. *For any family X_i, $i \in I$, of open sets, their union $\bigcup_{i\in I} X_i$ is an open set. For $I = \emptyset$, we define the union as $\emptyset$.*

Proof. 1. As for $\emptyset$, every statement of the form $\forall x \in X \ldots$ holds if $X = \emptyset$. As for M, every ball in M is a subset of M.

2. Let X and Y be open subsets of M and let $a \in X \cap Y$ (if $X \cap Y = \emptyset$, the claim holds by part 1). There exist real numbers $r, r' > 0$ such that $B(a, r) \subset X$ and $B(a, r') \subset Y$. We set $r'' := \min(r, r')$ (> 0). Since $B(a, r'')$ is contained in both balls $B(a, r)$ and $B(a, r')$ (and it is equal to one of them), we have $B(a, r'') \subset X \cap Y$.

3. Let $a \in \bigcup_{i\in I} X_i$ (if $\bigcup_{i\in I} X_i = \emptyset$, the claim holds by part 1). Then $a \in X_j$ for some $j \in I$. Thus $B(a, r) \subset X_j$ for some $r > 0$ and we have $B(a, r) \subset \bigcup_{i\in I} X_i$. □

Exercise 16 *Why is it true that "every statement of the form $\forall x \in \emptyset \ldots$ holds"?*

Exercise 17 *Show that any finite intersection of open sets is an open set, but that this in general does not hold for infinite intersections.*

We use the next identities to deduce from the properties of open sets the dual properties of closed sets.

Exercise 18 *Prove de Morgan identities. If U is a set and $A_i \subset U$ with $i \in I$ is any family of sets, then*

$$U \setminus \bigcup_{i\in I} A_i = \bigcap_{i\in I}(U \setminus A_i) \ \text{ and } \ U \setminus \bigcap_{i\in I} A_i = \bigcup_{i\in I}(U \setminus A_i)\,.$$

Corollary 19 *Let $M = \langle M, d\rangle$ be a metric space. The family of closed sets in M has the following properties.*

1. *The sets $\emptyset$ and M are closed.*
2. *If X and Y are closed sets, then their union $X \cup Y$ is closed.*
3. *For any family X_i, $i \in I$, of closed sets, their intersection $\bigcap_{i\in I} X_i$ is a closed set. For $I = \emptyset$, we define the intersection as M.*

Proof. 1. This follows from part 1 of Proposition 15 and from the definition of closed sets because $\emptyset$ and M are complements of one another.

2 and 3. These follow from the definition of closed sets and from parts 2 and 3 of Proposition 15 by means of de Morgan identities. □

Open and closed sets are sometimes viewed as opposites or as mutually exclusive, but this is an imprecise view. In every metric space M there are at least two sets that are both open and closed, namely $\emptyset$ and M. Such sets are called *clopen* and they play an important role in the definition of connected spaces, which we meet later.

Exercise 20 *Show that any finite union of closed sets is a closed set, but that this in general does not hold for infinite unions.*

Continuous maps between metric spaces

Let $M = \langle M, d\rangle$ and $N = \langle N, e\rangle$ be metric spaces, let $X \subset M$, and let $f\colon X \to N$ be a map. We say that f is *continuous at a point* $b \in X$ if for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$f([B_M(b,\, \delta)]) \subset B_N(f(b),\, \varepsilon)\,.$$

We say that the map f is *continuous* if it is continuous at every point in X. In the definition, we use the following notation for functions. If A and B are sets, $g\colon A \to B$ is a map, and C is *any* set (not necessarily a subset of A, as is usually redundantly assumed), we define the *image of C by g* as the set

$$g[C] := \{g(x)\colon\ x \in A \cap C\} \ \ (\subset B)\,.$$

Exercise 21 *In this situation, with M, N, X, f, and b as above, we can view f also as a map $f_0 = f$ from M' to N, for the subspace $M' = M\,|X$. Show that f is continuous at b (in the above sense) iff f_0 is continuous at b.*

So we lose nothing if we restrict ourselves to continuous, everywhere defined maps between metric spaces.

Continuity of maps between spaces can be equivalently characterized by the preimages of open (or closed) sets. Like above, if A and B are sets, $g\colon A \to B$ is a map, and C is *any* set, we define the *preimage of C by g* as the set

$$g^{-1}[C] := \{x \in A\colon\ g(x) \in C\} \ \ (\subset A)\,.$$

Proposition 22 *Let $M = \langle M, d\rangle$ and $N = \langle N, e\rangle$ be metric spaces and let $f\colon M \to N$ be a map. Then f is continuous*

$$\iff f^{-1}[B] \textit{ is an open set in } M \textit{ for every open set } B \textit{ in } N\,.$$

Proof. Suppose that f is continuous, that $B \subset N$ is any open set in N, and that $b \in f^{-1}[B]$ is any point in the preimage. Since $f(b) \in B$, there is an $\varepsilon > 0$ such that $B_N(f(b), \varepsilon) \subset B$. Since f is continuous at b, there is a $\delta > 0$ such that

$$f[B_M(b,\, \delta)] \subset B_N(f(b),\, \varepsilon) \ \ (\subset B)\,.$$

Thus $f(x) \in B$ for every $x \in B_M(b, \delta)$, which means that $B_M(b, \delta) \subset f^{-1}[B]$ and $f^{-1}[B]$ $(\subset M)$ is an open set.

Suppose that the right-hand side of the equivalence holds, that $b \in M$ is any point in M, and that an $\varepsilon > 0$ is given. By Exercise 14, the ball $B_N(f(b), \varepsilon)$ is an open set in N. By the assumption, the preimage $P := f^{-1}[B_N(f(b), \varepsilon)]$ is an open set in M. Since $b \in P$, we can take a $\delta > 0$ such that $B_M(b, \delta) \subset P$. It follows that

$$f[B_M(b,\, \delta)] \subset B_N(f(b),\, \varepsilon)$$

and f is continuous at b. This holds for every $b \in M$ and f is continuous. □

Exercise 23 *The equivalence holds also with closed sets.*

For the third equivalent formulation of continuity, we need limits of sequences in metric spaces. Let $M = \langle M, d\rangle$ be a metric space, $b \in M$, and let $(a_n) \subset M$ be a sequence of points in it. If for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for every index $n \geq n_0$ we have

$$d(a_n,\, b) \leq \varepsilon\,,$$

we write $\lim_{n\to\infty} a_n = b$ or $\lim a_n = b$ or $a_n \to b$ and say that *the sequence* (a_n) *has limit* b.

Exercise 24 *Show that* $\lim_{n\to\infty} a_n = b \iff \lim_{n\to\infty} d(a_n,\, b) = 0$ *(these are two different kinds of limit).*

Exercise 25 *Show that limits are unique, i.e. if* $a, b \in M$ *and if* $(c_n) \subset M$ *is a sequence of points in a metric space* M *such that both* $\lim c_n = a$ *and* $\lim c_n = b$*, then* $a = b$*.*

Theorem 26 (Heine's definition) *Let* $f\colon M \to N$ *be a map between metric spaces. Then* f *is continuous at a point* $b \in M$ *iff* $\lim f(a_n) = f(b)$ *for every sequence* $(a_n) \subset M$ *such that* $\lim a_n = b$*.*

Proof. Suppose that f is continuous at b, that $(a_n) \subset M$ is a sequence converging to b, and that an $\varepsilon > 0$ is given. Then $f[B_M(b, \delta)] \subset B_N(f(b), \varepsilon)$ for some $\delta > 0$. Since $a_n \to b$, there is n_0 such that $a_n \in B_M(b, \delta)$ for every index $n \geq n_0$. Then $f(a_n) \in B_N(f(b), \varepsilon)$ for the same indices n. We see that $f(a_n) \to f(b)$.

Suppose that f is not continuous at b. This means that there is $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ we have

$$f[B_M(b,\, 1/n)] \not\subset B_N(f(b),\, \varepsilon)\,.$$

Using the *axiom of choice*, we select for every index n in $\mathbb{N}$ an element a_n in $B_M(b, 1/n)$ such that $f(a_n) \notin B_N(f(b), \varepsilon)$. In this way, we get a sequence $(a_n) \subset M$ such that $\lim a_n = b$, but the sequence $(f(a_n))$ does not have the limit $f(b)$. $\square$

Eduard Heine (1821–1881) was a German mathematician.

Exercise 27 *Learn in Wikipedia about the set-theoretic axiom of choice and its import for mathematics.*

We prove a characterization of closed sets in terms of limits of sequences.

Proposition 28 *A set $X \subset M$ in a metric space $\langle M, d \rangle$ is closed if and only if for every sequence of points $(a_n) \subset X$ that has a limit, the limit lies in X.*

Proof. Suppose that X is closed, $(a_n) \subset X$ has a limit $\lim a_n = b$, but that $b \in M \setminus X$. Since the last set is open, we have $B(b, r) \subset M \setminus X$ for some $r > 0$. By the definition of limit, $a_n \in B(b, r)$ and hence $a_n \notin X$ for every large n, which is a contradiction. We see that $b \in X$.

Suppose that X is not closed. Then the complement $M \setminus X$ is not open, which means that there exists a point $b \in M \setminus X$ such that for every $n \in \mathbb{N}$ we have

$$B(b,\, 1/n) \cap X \neq \emptyset\,.$$

Using the *axiom of choice*, we select for every n a point $a_n \in B(b, 1/n) \cap X$. In this way we get a sequence (a_n) such that $(a_n) \subset X$ and $\lim a_n = b$. However, $b \notin X$. $\square$

Complete sets and complete spaces

Let $M = \langle M, d \rangle$ be a metric space and $(a_n) \subset M$ be a sequence of points in it. If for every $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that $d(a_m, a_n) \leq \varepsilon$ for every two indices $m, n \geq n_0$, we say that the sequence (a_n) is *Cauchy*.

Definition 29 (completeness) *A set $X \subset M$ in a metric space $\langle M, d \rangle$ is complete if every Cauchy sequence $(a_n) \subset X$ has a limit in X. If the set M is complete, we say that the space M is complete.*

For example, the real line $\mathbb{R}$, viewed as the metric space $E_1 = \langle \mathbb{R}, |x - y| \rangle$, is a complete space. This follows from the theorem we know from the course *MA1* that a sequence $(a_n) \subset \mathbb{R}$ is Cauchy if and only if it has a (finite) limit.

Exercise 30 *Prove that discrete metric spaces (we defined them at the beginning) are complete.*

Proposition 31 *Let $M = \langle M, d \rangle$ be a complete metric space and let $X \subset M$. Then the set X is complete if and only if it is closed.*

Proof. Suppose that X is complete and that $(a_n) \subset X$ is a sequence with a limit $\lim a_n = b$ $(\in M)$. Then (a_n) is a Cauchy sequence (by Exercise 32). Since X is complete and limits are unique, $b \in X$. By Proposition 28, X is closed.

Suppose that X is not complete. Then there is a Cauchy sequence $(a_n) \subset X$ such that (a_n) does not have a limit or it has a limit outside X. Since we are in a complete space M, the latter is true. By Proposition 28, X is not closed. $\square$

Exercise 32 *If a sequence in a metric space has a limit, then the sequence is Cauchy.*

Exercise 33 *Let A and B be complete sets in a metric space. Is $A \cap B$ complete? Is $A \cup B$ complete?*

References

1. Copson, E. T. *Metric spaces.* Cambridge Tracts in Mathematics and Mathematical Physics, No. 57 Cambridge University Press, London, 1968. v+143 pp.