Mathematical Analysis 1

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To the memory of
Jiří Matoušek (1963–2015)

Introduction

This textbook of mathematical analysis contains ?? chapters.

In Prague, February–??? 2022 Martin Klazar

Contents

Chapter 1

Lecture 2. Theorems on existence of limits of sequences

In the extended version of Lecture 2 we introduce arithmetic of infinities and define neighborhoods of points and infinities. Then we define simultaneously finite and infinite limits of real sequences and compute some, for example the limit of the n-th root of n. We prove that monotonous and quasi-monotonous sequences have limits and conclude this chapter by the proof of the Bolzano–Weierstrass theorem that every bounded sequence has a convergent subsequence, and the proof of the theorem on the Cauchy condition that a sequence converges iff it is Cauchy.

• *Review.* Recall the logical and set-theoretic notation introduced in lecture 1, recall what is $\mathbb{R}$ and recall the natural numbers $\mathbb{N} = \{1, 2, \dots\}$. We denote by i, j, k, l, m, m_0, m_1, $\dots$, n, n_0, n_1, $\dots$ natural numbers and a, b, c, d, e, δ, ε and θ, possibly with indices, are real numbers. Always $\delta, \varepsilon, \theta > 0$ and we think of them as close to zero. Recall what is a real sequence $(a_n) \subset \mathbb{R}$. Letters M and N usually denote sets of real numbers.

Exercise 1.1 *Read negated claim*

$$\forall\, \varepsilon\; \exists\, \delta\; \forall\, a,\, b \in M\colon |a - b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon\ .$$

• *Arithmetic of infinities.* So that we can define infinite limits we add to $\mathbb{R}$ the *infinities* $+\infty$ and $-\infty$. We get the *extended real line*

$$\mathbb{R}^* := \mathbb{R} \cup \{+\infty, -\infty\}\ .$$

We compute with infinities as follows.

$$\begin{aligned}
A \in \mathbb{R} \cup \{\pm\infty\} \quad &\Rightarrow \quad A + (\pm\infty) = \pm\infty + A := \pm\infty\ , \\
A \in (0,\, +\infty) \cup \{+\infty\} \quad &\Rightarrow \quad A \cdot (\pm\infty) = (\pm\infty) \cdot A := \pm\infty\ , \\
A \in (-\infty,\, 0) \cup \{-\infty\} \quad &\Rightarrow \quad A \cdot (\pm\infty) = (\pm\infty) \cdot A := \mp\infty \ \text{ and} \\
a \in \mathbb{R} \quad &\Rightarrow \quad \frac{a}{\pm\infty} := 0
\end{aligned}$$

where on each line we take only upper or only lower sings. Also, $-(\pm\infty) := \mp\infty$, $-\infty < a$ and $a < +\infty$ for every a, and $-\infty < +\infty$. Subtraction of an element $A \in \mathbb{R}^*$ reduces to adding $-A$. Division by an $a \neq 0$ reduces to multiplication by $1/a$. All remaining values of arithmetic operations with infinities, i.e., ($A \in \mathbb{R}^*$)

$$\frac{A}{0},\ (\pm\infty) + (\mp\infty),\ 0 \cdot (\pm\infty),\ (\pm\infty) \cdot 0,\ \frac{\pm\infty}{\pm\infty} \ \text{ and } \ \frac{\pm\infty}{\mp\infty}\ ,$$

are undefined, these are so called *indeterminate expressions*. The elements of $\mathbb{R}^*$ are usually denoted by A, B, K and L.

Exercise 1.2 *Compute* $\frac{-\infty}{-2}$, $(-\infty) - (+\infty)$, $-\infty + 10$ *and* $\frac{\pm\infty}{0}$.

Exercise 1.3 *Show that* $(\mathbb{R}^*, <)$ *is a linear order.*

Exercise 1.4 *Show that in* $(\mathbb{R}^*, <)$ *every subset* $X \subset \mathbb{R}^*$ *has both infimum and supremum — this is a nice property of the extended real line. Find the values of* $\inf(\emptyset)$, $\inf(\mathbb{R})$, $\sup(\{-\infty\})$ *and* $\sup(\mathbb{Z})$.

• *Neighborhoods of points and infinities.* Let us recall notation for real intervals:

$$(a,\, b] = \{x \in \mathbb{R} \mid a < x \le b\},\ (-\infty,\, a) = \{x \in \mathbb{R} \mid x < a\}$$

etc. One can still encounter notation with reversed brackets, where the above intervals are written as $]a, b]$ and $(-\infty, a[$, respectively.

Definition 1.5 (neighborhoods) *An* ε*-neighborhood of* $b \in \mathbb{R}$ *and a deleted* ε*-neighborhoods of* b *is defined as*

$$U(b,\, \varepsilon) := (b - \varepsilon,\, b + \varepsilon) \ \text{ and } \ P(b,\, \varepsilon) := (b - \varepsilon,\, b) \cup (b,\, b + \varepsilon)\ ,$$

respectively, so that $P(b, \varepsilon) = U(b, \varepsilon) \setminus \{b\}$. *Similarly,* ε*-neighborhoods of infinities are*

$$U(-\infty,\, \varepsilon) := (-\infty,\, -1/\varepsilon) \ \text{ and } \ U(+\infty,\, \varepsilon) := (1/\varepsilon,\, +\infty)\ .$$

We set $P(\pm\infty, \varepsilon) := U(\pm\infty, \varepsilon)$.

The main property of neighborhoods is that if $V, V' \in \{U, P\}$ then

$$A,\, B \in \mathbb{R}^*,\ A < B \Rightarrow \exists\, \varepsilon :\ V(A,\, \varepsilon) < V'(B,\, \varepsilon)\ ,$$

meaning that $a < b$ for every $a \in V(A, \varepsilon)$ and every $b \in V'(B, \varepsilon)$. In particular, $A \neq B \Rightarrow \exists\, \varepsilon :\ V(A, \varepsilon) \cap V'(B, \varepsilon) = \emptyset$. Next we give some more properties of neighborhoods.

Exercise 1.6 *Show that every $U(A, \varepsilon)$ is a convex set of real numbers: if $a < b < c$ with $a, c \in U(A, \varepsilon)$ then $b \in U(A, \varepsilon)$ too. Show that no deleted neighborhood $P(a, \varepsilon)$ is convex.*

Exercise 1.7 *Show that for any $V \in \{U, P\}$ and any $\delta \le \varepsilon$ one has that*

$$V(A,\, \delta) \subset V(A,\, \varepsilon)\ .$$

Exercise 1.8 *Show that for any a and any A,*

$$\bigcap_{\varepsilon>0} U(a,\, \varepsilon) = \{a\} \quad \textit{and} \quad \bigcap_{\varepsilon>0} U(\pm\infty,\, \varepsilon) = \bigcap_{\varepsilon>0} P(A,\, \varepsilon) = \emptyset\ .$$

• *Limits of sequences.* If it is not said else, $(a_n), (b_n), (c_n) \subset \mathbb{R}$ denote real sequences. The next definition belongs to the fundamental ones in analysis and the whole mathematics.

Definition 1.9 (limit of a sequence) *Let $(a_n) \subset \mathbb{R}$ be a sequence and let $L \in \mathbb{R}^*$. If*

$$\forall\, \varepsilon\ \exists\, n_0 :\ n \ge n_0 \Rightarrow a_n \in U(L,\, \varepsilon)\ ,$$

we write that $\lim a_n = L$ *and say that the sequence (a_n) has the limit L.*

$L \in \mathbb{R}$ are *finite* limits and $L = \pm\infty$ are infinite limits. Sequences with finite limits *converge*, else they *diverge*. Shortly we prove that limits, if they exist, are unique. A finite limit $\lim a_n = a$ means that for every real (and no matter how small) $\varepsilon > 0$ there exists an index $n_0 \in \mathbb{N}$ such that for every index $n \in \mathbb{N}$ from n_0 on the distance between a_n and a is less than ε:

$$|a_n - a| < \varepsilon\ .$$

Infinite $\lim a_n = -\infty$ means that for every (negative) $c \in \mathbb{R}$ there exists an index n_0 such that for every index n from n_0 on,

$$a_n < c\ .$$

Similarly, with reversed inequality, for the limit $+\infty$. Variant notation for limits which we will use is

$$\lim_{n\to\infty} a_n = L \ \text{ and } \ a_n \to L\ .$$

The simplest convergent sequence is an *eventually constant* sequence (a_n) with $a_n = a$ for every $n \ge n_0$, then of course $\lim a_n = a$. A popular image of limits by which “a sequence approaches its limit arbitrarily closely but never reaches it exactly (possibly only in infinity)” is a poetic one but is incorrect.

Exercise 1.10 *Let (a_n) and (b_n) satisfy $a_n = b_n$ for every $n \ge n_0$. Show that then*

$$\lim a_n = A \Rightarrow \lim b_n = A\ .$$

Proposition 1.11 (uniqueness of limits) *Limits of sequences are unique,*

$$\lim a_n = K \wedge \lim a_n = L \Rightarrow K = L\ .$$

Proof. Let $\lim a_n = K$, $\lim a_n = L$ and an ε be given. By the definition 1.9 there is an n_0 such that $n \geq n_0 \Rightarrow a_n \in U(K, \varepsilon)$ and $a_n \in U(L, \varepsilon)$. Thus $\forall\, \varepsilon :\ U(K, \varepsilon) \cap U(L, \varepsilon) \neq \emptyset$. But then by the main property of neighborhoods mentioned above, $K = L$. □

• *Two limits.* We show that $\lim \frac{1}{n} = 0$. This is clear, for every ε and every $n \geq n_0 := 1 + \lceil 1/\varepsilon \rceil$,

$$0 < \frac{1}{n} \leq \underbrace{\frac{1}{1 + \lceil 1/\varepsilon \rceil}}_{> 1/\varepsilon} < \frac{1}{1/\varepsilon} = \varepsilon \rightsquigarrow 1/n \in U(0,\, \varepsilon)\ .$$

Here $\lceil a \rceil \in \mathbb{Z}$ denotes the *upper integral part* of a number a, the least $v \in \mathbb{Z}$ such that $v \geq a$. Similarly, the *lower integral part* $\lfloor a \rfloor$ of a number a is the largest $v \in \mathbb{Z}$ such that $v \leq a$. Our second example is that

$$\sqrt[3]{n} - \sqrt{n} \to -\infty\ .$$

Indeed, for any given $c < 0$ for every $n \geq n_0 > \max(4c^2, 2^6)$ it holds that

$$\overbrace{\sqrt[3]{n} - \sqrt{n}}^{\text{non-trivial}} = \overbrace{n^{1/2} \cdot \underbrace{(n^{-1/6} - 1)}_{n > 2^6\ \Rightarrow\ \cdots < -1/2} \quad < \quad \underbrace{-n^{1/2}}_{\cdots < -2|c|}}^{\text{trivial}} /2 < -2|c|/2 = c\ .$$

One does not have to find an optimum value of the index n_0 in terms of ε or c. This can be done easily only in the simplest cases like $\lim \frac{1}{n}$, otherwise it may be hard to do. But any value of n_0 suffices such that if $n \geq n_0$ then the membership $a_n \in U(L, \varepsilon)$ in the definition of limit holds. To find such n_0 it helps to have some skill in manipulating inequalities and estimates.

Exercise 1.12 *Compute the limit*

$$\lim_{n \to \infty} \frac{\sqrt[3]{n} - \sqrt{n}}{\sqrt[4]{n}}\ .$$

• *Subsequences of sequences.* A subsequence arises by omitting some terms of the given sequence so that still an infinite sequence remains. Formal definition follows. In exercise 1.17 we introduce weak subsequences.

Definition 1.13 (subsequence) *We say that (b_n) is a subsequence of (a_n) if there exists a sequence of natural numbers $m_1 < m_2 < \ldots$ such that for every n,*

$$b_n = a_{m_n}\ .$$

We denote this relation between (b_n) and (a_n) by $(b_n) \preceq (a_n)$.

The relation $\preceq$ on the set of real sequences is reflexive and transitive. It is easy to find sequences (a_n) a (b_n) such that $(a_n) \preceq (b_n)$ and $(b_n) \preceq (a_n)$ but $(a_n) \neq (b_n)$. The above definition extends without change to sequences $(a_n) \subset X$ in arbitrary sets X.

Proposition 1.14 ($\preceq$ preserves limits) *Let* $(b_n) \preceq (a_n)$ *and* $\lim a_n = L \in \mathbb{R}^*$. *Then* $\lim b_n = L$ *too.*

Proof. It follows at once from the definitions 1.9 and 1.13, the sequence (m_n) in the last definition satisfies that $m_n \geq n$ for every n. $\square$

The following useful proposition holds and later we prove part 1 of it. Proofs of parts 2 and 3 are left for exercise 1.16.

Proposition 1.15 (on subsequences) *Let* $(a_n) \subset \mathbb{R}$ *and let* $A \in \mathbb{R}^*$. *The following hold.*

1. *There is a sequence* (b_n) *such that* $(b_n) \preceq (a_n)$ *and* (b_n) *has a limit.*
2. *The sequence* (a_n) *does not have a limit* $\iff$ (a_n) *has two subsequences with different limits.*
3. *It is not true that* $\lim a_n = A \iff$ *there is a sequence* (b_n) *such that* $(b_n) \preceq (a_n)$ *and* (b_n) *has a limit different from* A.

So by part 2 one can always refute that a sequence in question has a limit by exhibiting two subsequences with different limits. For example,

$$(a_n) := ((-1)^n) = (-1,\, 1,\, -1,\, 1,\, -1,\, \dots)$$

does not have a limit because $(1, 1, \dots) \preceq (a_n)$ and $(-1, -1, \dots) \preceq (a_n)$.

Exercise 1.16 *Prove parts 2 and 3 in the previous proposition.*

Exercise 1.17 (weak subsequences) *We say that* (b_n) *is a weak subsequence of* (a_n) *if there is a sequence* $(m_n) \subset \mathbb{N}$ *such that*

$$\lim m_n = +\infty \wedge \forall\, n \colon b_n = a_{m_n}\ .$$

We denote this relation between (b_n) *and* (a_n) *by* $(b_n) \preceq^* (a_n)$. *Prove a generalization of Proposition 1.14 that*

$$(b_n) \preceq^* (a_n) \wedge \lim a_n = L \Rightarrow \lim b_n = L\ .$$

• *The limit of n-th root of n.* Limits of sequences can be divided in "trivial" and "non-trivial" ones. The former case occurs when no two growths (usually to infinity) in the expression do not fight each other, else we have the latter case. To put it in other words, a nontrivial limit is that of an indeterminate expression, else the limit is trivial. For instance, the $\lim\,(2^n + 3^n)$ and $\lim \frac{4}{5n-3}$

are trivial limits whereas the limits $\lim\,(2^n - 3^n)$ and $\lim\,\frac{4n+7}{5n-3}$ are non-trivial. Often one can compute a non-trivial limit by transforming it algebraically in a trivial limit, like in the above example with $\sqrt[3]{n} - \sqrt{n}$:

$$\sqrt[3]{n} - \sqrt{n} \rightsquigarrow +\infty - (+\infty) = ? \text{ but } n^{1/2}(n^{-1/6} - 1) \rightsquigarrow (+\infty) \cdot (0 - 1) = -\infty \ .$$

The next limit of $n^{1/n}$ is non-trivial because the base $n \to +\infty$ but the exponent $1/n \to 0$ and $(+\infty)^0$ is a new indeterminate expression. We show that the exponent prevails and $n^{1/n} \to 1$.

Proposition 1.18 ($n^{1/n} \to 1$) *It holds that*

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \sqrt[n]{n} = 1 \ .$$

Proof. Always $n^{1/n} \geq 1$. If $n^{1/n} \not\to 1$, there is a number $c > 0$ and a sequence $2 \leq n_1 < n_2 < \dots$ such that for every i one has that $n_i^{1/n_i} > 1 + c$. By the binomial theorem (exercise 1.19) one has for every i that

$$\begin{aligned} n_i &> (1+c)^{n_i} = \sum_{j=0}^{n_i} \binom{n_i}{j} c^j = 1 + \binom{n_i}{1} c + \binom{n_i}{2} c^2 + \dots + \binom{n_i}{n_i} c^{n_i} \\ &\geq \frac{n_i(n_i - 1)}{2} \cdot c^2 \end{aligned}$$

and so, for every i,

$$n_i > \frac{n_i(n_i-1)}{2} \cdot c^2 \rightsquigarrow 1 + \frac{2}{c^2} > n_i \ .$$

The sequence $n_1 < n_2 < \dots$ is not bounded from above and we have a contradiction. $\Box$

Exercise 1.19 (Binomial Theorem) *Give some proof of it. The theorem says that for every two formal variables x and y and every exponent $n \in \mathbb{N}_0$ one has the identity*

$$(x+y)^n = \underbrace{(x+y) \cdot (x+y) \cdot \ldots \cdot (x+y)}_{n \text{ factors}} = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j} \ .$$

Here, for every $j, n \in \mathbb{N}$,

$$\binom{n}{j} = \frac{n(n-1)\dots(n-j+1)}{j!}$$

is so called binomial coefficient, where $j! = 1 \cdot 2 \cdot \ldots \cdot j$ is the factorial function. For every $n \in \mathbb{N}_0$ we set $\binom{n}{0} := 1$. Also every power with exponent 0 is defined as 1.

• *Robust properties of sequences.* Below we present four theorems (1.25, 1.28, 1.31 and 1.35) on existence of limits of real sequences. Exercise 1.10 shows that the limit of a sequence (its existence and value) is insensitive to changes of only finitely many terms in the sequence. Hence properties of sequences ensuring existence of limits should be also robust in the same sense, should resist changes of only finitely many terms in the sequences. Here is the formal definition.

Definition 1.20 (robustness) *A robust property of sequences of real numbers is any set of real sequences*

$$X \subset \mathbb{R}^{\mathbb{N}}\ ,$$

such that if $(a_n) \in X$ *and* $(b_n) \subset \mathbb{R}$ *satisfies* $b_n = a_n$ *for every* $n \geq n_0$ *then* $(b_n) \in X$.

Definition 1.21 ((un)bounded sequences) *A sequence* $(a_n) \subset \mathbb{R}$ *is*

- *bounded from above if there is a* c *such that* $a_n < c$ *for every* n*,*
- *unbounded from above if such* c *does not exist,*
- *bounded from below if there is a* c *such that* $a_n > c$ *for every* n*,*
- *unbounded from below if such* c *does not exist,*
- *bounded if it is bounded both from below and from above.*

Exercise 1.22 *Show that each of the five properties of sequences in the previous definition is robust.*

• *Monotone sequences.* Often one can see the next theorem on monotone sequences stated only for sequences (a_n) monotone for every n. But this is not a robust property. In our four theorems on existence of limits of sequences we employ only robust properties.

Definition 1.23 (monotonicíty) *A sequence* (a_n) *is*

- *non-decreasing if* $a_n \leq a_{n+1}$ *for every* n*,*
- *non-decreasing from* n_0 *if* $a_n \leq a_{n+1}$ *for every* $n \geq n_0$ *for some* n_0*,*
- *non-increasing if* $a_n \geq a_{n+1}$ *for every* n*,*
- *non-increasing from* n_0 *if* $a_n \geq a_{n+1}$ *for every* $n \geq n_0$ *for some* n_0*,*
- *monotonous (monotone) if it is non-decreasing or non-increasing,*
- *monotonous (monotone) from* n_0 *if it is non-decreasing from* n_0 *or non-increasing from* n_0 *for some* n_0*.*

Strict inequalities $a_n < a_{n+1}$*, resp.* $a_n > a_{n+1}$*, yield (strictly) increasing, resp. (strictly) decreasing, sequences.*

Exercise 1.24 *Which of the six properties in the previous definition is robust?*

Theorem 1.25 (on monotone sequences). *Every real sequence (a_n) that is monotone from n_0 has a limit. If (a_n) is non-decreasing from n_0 then*

$$\lim_{n\to\infty} a_n = \begin{cases} \sup(\{a_n \mid n \ge n_0\}) \in \mathbb{R} & \ldots \quad (a_n) \text{ is bounded from above and} \\ +\infty & \ldots \quad (a_n) \text{ is unbounded from above.} \end{cases}$$

If (a_n) is non-increasing from n_0 then

$$\lim_{n\to\infty} a_n = \begin{cases} \inf(\{a_n \mid n \ge n_0\}) \in \mathbb{R} & \ldots \quad (a_n) \text{ is bounded from below and} \\ -\infty & \ldots \quad (a_n) \text{ is unbounded from below.} \end{cases}$$

Proof. We consider only the first case of a sequence non-decreasing from n_0, the other case is similar. When (a_n) is unbounded from above then for any given c there exists an m such that $a_m > \max(c, a_1, a_2, \ldots, a_{n_0})$. Thus $a_m > c$ and $m > n_0$, and therefore for every $n \ge m$,

$$a_n \ge a_{n-1} \ge \cdots \ge a_m > c \rightsquigarrow a_n > c$$

and $a_n \to +\infty$.

For (a_n) bounded from above we set $s := \sup(\{a_n \mid n \ge n_0\})$. Let an $\varepsilon > 0$ be given. By the definition of supremum there exists an $m \ge n_0$ such that $s - \varepsilon < a_m \le s$. Therefore for every $n \ge m$,

$$s - \varepsilon < a_m \le a_{m+1} \le \cdots \le a_{n-1} \le a_n \le s \rightsquigarrow s - \varepsilon < a_n \le s$$

and $a_n \to s$. $\square$

• *Quasi-monotonous (quasi-monotone) sequences.* This is a supplementary passage.

Definition 1.26 (quasi-monotonicity) *A sequence (a_n) is quasi-monotone from n_0 if*

$$n \ge n_0 \Rightarrow \text{every set } \{m \mid a_m < a_n\} \text{ is finite}$$

or

$$n \ge n_0 \Rightarrow \text{every set } \{m \mid a_m > a_n\} \text{ is finite}\,,$$

for some n_0.

Any sequence monotonous from n_0 is quasi-monotonous from the same n_0.

Exercise 1.27 *Give an example of a sequence that is not monotone from n_0 for any n_0 but is quasi-monotone from some n_0.*

The next theorem uses parameters lim sup and lim inf of sequences that are always defined and may attain values $\pm\infty$, and that will be introduced in the next lecture.

Theorem 1.28 (on quasi-monotone sequences). *Every sequence* $(a_n) \subset \mathbb{R}$ *that is quasi-monotone from* n_0 *has a limit. If* (a_n) *satisfies the first, resp. the second, condition in Definition 1.26 then*

$$\lim\, a_n = \limsup a_n \in \mathbb{R}^*, \;\; \textit{resp.} \;\; \lim\, a_n = \liminf a_n \in \mathbb{R}^* \,.$$

Proof. We only consider the case that (a_n) satisfies the first condition for some n_0, the other case is similar. Let (a_n) be unbounded from above and a c be given. Thus there is an $m \ge n_0$ such that $a_m > c$. By the first condition there is a k such that $a_n \ge a_m > c$ for every $n \ge k$. Hence $a_n \to +\infty = \limsup\, a_n$. Let (a_n) be bounded from above, $s := \limsup\, a_n \in \mathbb{R}$ and let an ε be given. By the definition of $\limsup\, a_n$, in

$$s - \varepsilon < a_m < s + \varepsilon$$

the first inequality holds for infinitely many m and the second one for every $m \ge m_0$. By the first condition there is a k such that $s - \varepsilon < a_n < s + \varepsilon$ holds for every $n \ge k$. Thus $a_n \to s = \limsup\, a_n$. □

Quasi-monotone sequences, for which $n_0 = 1$, were introduced by the English mathematician *Godfrey H. Hardy (1877–1947).*

• *The Bolzano–Weierstrass theorem.* The following auxiliary result is of interest by itself.

Proposition 1.29 (monotone subsequence) *Every sequence of real numbers has a monotone subsequence.*

Proof. For a given sequence (a_n) we define the set

$$M := \{n \mid \forall\, m : \;\; n \le m \Rightarrow a_n \ge a_m\} \,.$$

If it is infinite, $M = \{m_1 < m_2 < \dots\}$, then (a_{m_n}) is a non-increasing subsequence. If M is finite, we take a number $m_1 > \max(M)$. Then $m_1 \notin M$ and there is a number $m_2 > m_1$ such that $a_{m_1} < a_{m_2}$. Since $m_2 \notin M$, there is a number $m_3 > m_2$ such that $a_{m_2} < a_{m_3}$. And so on, we have a non-decreasing, in fact strictly increasing, subsequence (a_{m_n}). □

The theorem on monotone sequences and the previous proposition have as immediate corollaries the following two results. The first one is part 1 of Proposition 1.15.

Corollary 1.30 (subsequence with limit) *Every sequence of real numbers has a subsequence that has a limit.*

Theorem 1.31 (Bolzano–Weierstrass theorem) *Every bounded sequence of real numbers has a convergent subsequence.*

Karl Weierstrass (1815–1897) was a German mathematician, he was a "father of modern mathematical analysis". The priest, philosopher and mathematician *Bernard Bolzano (1781–1848)* had Italian, German and Czech roots. A street near the main railway station in Prague is named after him, in Celetná street a plaque commemorates him and his grave can be found in Olšanské hřbitovy (cemetery).

Exercise 1.32 *Prove this version of the Bolzano–Weierstrass theorem: for every real numbers $a \le b$, every sequence*

$$(a_n) \subset [a, b]$$

has a convergent subsequence with limit in the interval $[a, b]$.

• *The Cauchy condition.* The terms in Cauchy sequences get arbitrarily close each to the other as their indices grow. Formal definition follows.

Definition 1.33 (Cauchy sequences) *A sequence $(a_n) \subset \mathbb{R}$ is Cauchy if*

$$\forall \varepsilon \, \exists n_0 : \ m, n \ge n_0 \Rightarrow |a_m - a_n| < \varepsilon \ ,$$

i.e., $a_m \in U(a_n, \varepsilon)$.

Clearly, this is a robust property of sequences.

Exercise 1.34 *Prove that every Cauchy sequence of real numbers is bounded.*

Theorem 1.35 (Cauchy condition) *Let $(a_n) \subset \mathbb{R}$. Then (a_n) is convergent if and only if (a_n) is Cauchy.*

Proof. The implication $\Rightarrow$. Let $\lim a_n = a$ and an ε be given. Then there is an n_0 such that $n \ge n_0 \Rightarrow |a_n - a| < \varepsilon/2$. Thus

$$m, n \ge n_0 \Rightarrow |a_m - a_n| \le |a_m - a| + |a - a_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and (a_n) is a Cauchy sequence. We used the expression $a_m - a_n = (a_m - a) + (a - a_n)$ and the triangle inequality $|c + d| \le |c| + |d|$.

The implication $\Leftarrow$. Let (a_n) be a Cauchy sequence. As we know, (a_n) is bounded and therefore by the Bolzano–Weierstrass theorem it has a convergent subsequence (a_{m_n}) with a limit a. So for a given ε we have an n_0 such that $n \ge n_0 \Rightarrow |a_{m_n} - a| < \varepsilon/2$ and that $m, n \ge n_0 \Rightarrow |a_m - a_n| < \varepsilon/2$. Always $m_n \ge n$, hence

$$n \ge n_0 \Rightarrow |a_n - a| \le |a_n - a_{m_n}| + |a_{m_n} - a| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

and $a_n \to a$. □

Interestingly, the French mathematician *Augustin-Louis Cauchy (1789–1857)* also lived in Prague for some time, in political exile in 1833–1838.

Exercise 1.36 (triangle inequality) *Prove that for arbitrary real numbers* $a_1, a_2, \ldots, a_n$,

$$|a_1 + a_2 + \cdots + a_n| \leq |a_1| + |a_2| + \cdots + |a_n| \, .$$

Exercise 1.37 *Give an example of a Cauchy sequence* $(a_n) \subset \mathbb{Q}$ *with irrational limit. The previous theorem therefore does not hold in the ordered field* $\mathbb{Q}$ *— it happens because* $\mathbb{Q}$ *is not complete.*

Exercise 1.38 *Where was completeness of the real numbers used in the previous proof?*

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The numbers of pages in *slanted font* refer to basic definitions and theorems with proofs. By $^+$ we denote the theorems and proofs in supplementary chapters.