LECTURE 8, 4/6/2022
MEAN VALUE THEOREMS AND THEIR COROLLARIES

- *Mean value theorems.* We present three of them.

Theorem 1 (Rolle's) *Let $a < b$ be real numbers and $f\colon [a,b] \to \mathbb{R}$ with $f(a) = f(b)$ be a continuous function that has finite or infinite derivative at each point of the interval (a,b). Then*

$$\exists\, c \in (a,\, b)\colon\, f'(c) = 0\ .$$

Proof. If f is constant, that is $f(x) = f(a) = f(b)$ for every $x \in [a,b]$, then $f'(x) = 0$ for every $x \in (a,b)$. Let f not be constant and $f(x) > f(a) = f(b)$ for some $x \in (a,b)$, the case with $f(x) < f(a) = f(b)$ is treated similarly. According to the principle of minimum and maximum (in lecture 6), the function f attains its greatest value in some $c \in [a,b]$. Clearly, $c \in (a,b)$. By the assumption about derivatives and Theorem 4 in the last lecture, $f'(c) = 0$. □

Theorem 2 (Lagrange's) *Let $a < b$ be real numbers and $f\colon [a,b] \to \mathbb{R}$ be a continuous function that has finite or infinite derivative at each point of the interval (a,b). Then*

$$\exists\, c \in (a,\, b)\colon\, f'(c) = \frac{f(b) - f(a)}{b - a}\ .$$

Proof. Consider the function

$$g(x) := f(x) - (x - a) \cdot \frac{f(b) - f(a)}{b - a}\colon [a,\, b] \to \mathbb{R}\ .$$

It satisfies the assumptions of Rolle's theorem, especially $g(a) = g(b) = f(a)$, therefore

$$0 = g'(c) = f'(c) - (f(b) - f(a))/(b - a)$$

for some $c \in (a, b)$ and we are done. $\square$

Geometrically, this theorem says that under the given assumptions there is always a tangent to G_f at some point $(c, f(c))$, $c \in (a, b)$, which is parallel to the secant $\kappa(a, f(a), b, f(b))$.

Theorem 3 (Cauchy's) *Let $a < b$ be real numbers and $f, g\colon [a, b] \to \mathbb{R}$ with $g(b) \neq g(a)$ be continuous functions that have derivative at each point of the interval (a, b). Derivatives of the function f may be infinite, but derivatives of the function g have to be finite. Then*

$$\exists\, c \in (a,\, b)\colon\, f'(c) = \frac{f(b) - f(a)}{g(b) - g(a)} \cdot g'(c)\,.$$

Proof. Consider the function

$$h(x) := f(x) - (g(x) - g(a)) \cdot \frac{f(b) - f(a)}{g(b) - g(a)} \colon [a,\, b] \to \mathbb{R}\,.$$

It satisfies the assumptions of Rolle's theorem, especially $h(a) = h(b) = f(a)$, therefore

$$0 = h'(c) = f'(c) - g'(c) \cdot (f(b) - f(a))/(g(b) - g(a))$$

for some $c \in (a, b)$ and we are done. $\square$

• *Derivatives and monotonicity of functions.* A non-negative (resp. non-positive) derivative means that the original function does not decrease (resp. does not increase). A positive (resp. negative) derivative means that the original function increases (resp. decreases). The following theorem gives details. For any set $M \subset \mathbb{R}$ we denote by $M^0 := \{a \in M \mid \exists\, \delta \colon U(a,\delta) \subset M\}$ its *interior*. The interior of an interval I is the open interval $I^0 \subset I$ obtained from I by omitting the endpoints.

Theorem 4 (derivatives and monotonicity 1) *Let $I \subset \mathbb{R}$ be an interval and $f \colon I \to \mathbb{R}$ be a continuous function that has finite or infinite derivative at each point in the interior I^0 of I. Then the following hold.*

1. *$f' \ge 0$, resp. $f' \le 0$, on $I^0 \Rightarrow f$ is non-decreasing, resp. non-increasing, on I.*
2. *$f' > 0$, resp. $f' < 0$, on $I^0 \Rightarrow f$ is increasing, resp. decreasing, on I.*

Proof. Let $f' < 0$ on I^0 and $x < y$ be in I. By Theorem 2,

$$\frac{f(y) - f(x)}{y - x} = f'(z) < 0$$

for some $z \in (x, y) \subset I^0$. This inequality and $y - x > 0$ imply that $f(x) > f(y)$—f decreases on I. The other three cases in 1 and 2 are treated similarly. $\square$

The proof of the following proposition is similar to the proof of Theorem 4 in the last lecture and therefore we omit it.

Proposition 5 (derivative and monotonicity 2) *Let $a \in M \subset \mathbb{R}$, $f\colon M \to \mathbb{R}$ be a function and the one-sided derivatives below may be infinite. The following hold.*

1. *When a is a left limit point of M and $f'_-(a) < 0$, resp. $f'_-(a) > 0$, then there exists a δ such that*

$$f[P^-(a, \delta) \cap M] > \{f(a)\}, \text{ resp. } < \{f(a)\} .$$

2. *When a is a right limit point of M and $f'_+(a) < 0$, resp. $f'_+(a) > 0$, then there exists a δ such that*

$$f[P^+(a, \delta) \cap M] < \{f(a)\}, \text{ resp. } > \{f(a)\} .$$

Last time we calculated that $(|x|)'_-(0) = -1$ and $(|x|)'_+(0) = 1$. Thus, according to the previous proposition, the function $|x|$ has a strict local minimum in 0. Of course, this is clear even without any theory.

- *Extending derivatives by limits.*

Proposition 6 (extending derivatives) *Let $a, b \in \mathbb{R}$ with $a < b$, $f\colon [a, b) \to \mathbb{R}$ be a continuous function that has finite derivative on the interval (a, b) and let $\lim_{x\to a} f'(x) =: L \in \mathbb{R}^*$. Then*

$$f'_+(a) = L .$$

Proof. Let a, b, f and L be as stated, and let an ε be given. There exists a $\delta \le b - a$ such that $x \in P^+(a, \delta) \Rightarrow f'(x) \in U(L, \varepsilon)$. Let $x \in P^+(a, \delta)$ be arbitrary. According to Theorem 2, there exists a $y \in (a, x) \subset P^+(a, \delta)$ such that

$$\frac{f(x) - f(a)}{x - a} = f'(y) \in U(L,\, \varepsilon)\ .$$

Thus $f'_+(a) = L$. $\square$

A similar proposition holds for left derivatives.

• *l'Hospital's rule.* This is a method for calculating limits of ratios of functions $f(x)/g(x)$ leading to indefinite expressions $0/0$ and $\pm\infty/\pm\infty$.

Theorem 7 (l'Hospital's rule) *Let $A \in \mathbb{R}$. Let for some δ functions $f, g\colon P^+(A, \delta) \to \mathbb{R}$ have finite derivatives on $P^+(A, \delta)$, $g' \neq 0$ on $P^+(A, \delta)$, and let*

1. $\lim_{x \to A} f(x) = \lim_{x \to A} g(x) = 0$ *or*
2. $\lim_{x \to A} g(x) = \pm\infty$.

Then

$$\lim_{x \to A} \frac{f(x)}{g(x)} = \lim_{x \to A} \frac{f'(x)}{g'(x)}$$

if the last limit exists. This theorem also holds for left neighborhoods $P^-(A, \delta)$, ordinary neighborhoods $P(A, \delta)$ and for $A = \pm\infty$.

Proof. 1. Let $\lim_{x \to A} f(x) = \lim_{x \to A} g(x) = 0$, $\lim_{x \to A} \frac{f'(x)}{g'(x)} =: L \in \mathbb{R}^*$ and $A \in \mathbb{R}$. We define $f(A) = g(A) := 0$. A is a limit point of the definition domain of the fraction $f(x)/g(x)$: it is not possible that $g = 0$ on some $P^+(A, \theta)$, for then also $g' = 0$ on $P^+(A, \theta)$. We set

$$P^+_0(A,\, \delta) := \{x \in (A,\, A + \delta) \mid g(x) \neq 0\}\ .$$

By Theorem 3, there is a function $c\colon P_0^+(A,\delta) \to P^+(A,\delta)$ such that for every $x \in P_0^+(A,\delta)$,

$$c(x) \in (A,x) \ \text{ and } \ \frac{f(x)}{g(x)} = \frac{f(x)-f(A)}{g(x)-g(A)} = \frac{f'(c(x))}{g'(c(x))} \ .$$

Clearly, $\lim_{x\to A} c(x) = A$. Since $A \notin P^+(A,\delta)$, condition 1 in the theorem on limits of composite functions is satisfied. According to this theorem, we get that

$$\lim_{x\to A} \frac{f(x)}{g(x)} = \lim_{x\to A} \frac{f'(c(x))}{g'(c(x))} = \lim_{y\to A} \frac{f'(y)}{g'(y)} = L \ .$$

The proof for functions defined on $P^-(A,\delta)$ is similar. We reduce $P(A,\delta)$ to two one-sided neighborhoods. Finally, let $A = +\infty$, the case with $A = -\infty$ is treated similarly. By substituting $x := 1/y$ and using the theorem on limits of composite functions we reduce it to the limit at 0 and the definition domain $P^+(0,\delta)$:

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{y\to 0} \frac{f(1/y)}{g(1/y)}$$

and

$$\begin{aligned} \lim_{x\to+\infty} \frac{f'(x)}{g'(x)} &= \lim_{y\to 0} \frac{f'(1/y)}{g'(1/y)} = \lim_{y\to 0} \frac{f'(1/y)\cdot(-y^{-2})}{g'(1/y)\cdot(-y^{-2})} \\ &= \lim_{y\to 0} \frac{(f(1/y))'}{(g(1/y))'} \ , \end{aligned}$$

where the first equality holds due to the theorem on limits of composite functions and the last due to the formula for derivatives of composite functions.

2. Let $\lim_{x\to A} g(x) = \pm\infty$ and $\lim_{x\to A} \frac{f'(x)}{g'(x)} =: L \in \mathbb{R}^*$. We will prove this case later using integrals. $\square$

In the theorem, the slightly confusing notation $\lim_{x\to A^+}$ is sometimes used. But thanks to the definition domain $P^+(A,\delta)$ we can use the simpler ordinary limits $\lim_{x\to A}$.

We calculate by means of l'Hospital's rule a few limits. For example,

$$\begin{aligned}\lim_{x\to 0}\sqrt{x}\log x &= \lim_{x\to 0}\frac{(\log x)'}{(1/\sqrt{x})'} = \lim_{x\to 0}\frac{1/x}{(-1/2)x^{-3/2}} \\ &= -2\lim_{x\to 0}x^{1/2} = 0\ ,\end{aligned}$$

and more generally $\lim_{x\to 0}x^c\log x = 0$ for every $c>0$. Or

$$\begin{aligned}\lim_{x\to 0}\frac{x^2}{\cos x - 1} &= \lim_{x\to 0}\frac{(x^2)'}{(\cos x - 1)'} = \lim_{x\to 0}\frac{2x}{-\sin x} \\ &= -2\lim_{x\to 0}\frac{(x)'}{(\sin x)'} = -2\lim_{x\to 0}\frac{1}{\cos x} = -2\ .\end{aligned}$$

• *Higher order derivatives.* Definition domains of function will now mostly be open sets. Each point of such a set is its TLP.

Definition 8 ($f^{(n)}(x)$) *Let $M \subset \mathbb{R}$ be a nonempty open set and $f = f(x)\colon M \to \mathbb{R}$ be a function. For $n \in \mathbb{N}_0 = \{0, 1, \dots\}$ we define by induction a finite or infinite sequence of functions $f^{(n)}(x)\colon M \to \mathbb{R}$.*

1. *At the beginning we set $f^{(0)}(x) := f(x)$.*
2. *For $n > 0$, when the function $f^{(n-1)}(x)$ is defined and has finite derivative at each point $a \in M$, we define for each $a \in M$ the value of the n-th function as*

$$f^{(n)}(a) := (f^{(n-1)}(x))'(a) \ .$$

The function $f^{(n)}$ is called the order n derivative of the function f or the n-th derivative of f.

So the function $f^{(0)}$ is f itself and $f^{(1)}$ is its derivative f'. If $f^{(n-1)}\colon M \to \mathbb{R}$ is defined and has derivative at a point $b \in M$, finite or infinite, we still write

$$f^{(n)}(b) := (f^{(n-1)})'(b) \in \mathbb{R}^*$$

and call it the n-th derivative of the function f at the point b. The function $f^{(2)}$, the second derivative of f, is also denoted as f''. For example, for $M = \mathbb{R}$, $(x \sin x)'' = (\sin x + x \cos x)' = 2 \cos x - x \sin x$. Second derivatives can be used to justify the existence of extremes of functions.

Proposition 9 (f'' and extremes) *Suppose that $a \in M$, that $M \subset \mathbb{R}$ is an open set, and that $f\colon M \to \mathbb{R}$ is a function with finite $f'\colon M \to \mathbb{R}$, $f'(a) = 0$ and $f''(a) \in \mathbb{R}^*$, possibly infinite. Then the following hold.*

1. *$f''(a) > 0 \Rightarrow f$ has at a a strict local minimum.*
2. *$f''(a) < 0 \Rightarrow f$ has at a a strict local maximum.*

It is clear that the set M can be taken in the form $U(a, \delta)$.

Proof. We prove only part 1, for part 2 the argument is similar. So let $M = U(a, \delta)$, on $U(a, \delta)$ there exists finite f', $f'(a) = 0$ and $f''(a) > 0$. By Proposition 5 there exists a $\theta \le \delta$ such that $f' < f'(0) = 0$ on $P^-(a, \theta)$ and $f' > f'(0) = 0$ on $P^+(a, \theta)$. Let $x \in P^-(a, \theta)$ be arbitrary. By Theorem 2 there exists a $y \in (x, a) \subset P^-(a, \theta)$ such that

$$\frac{f(a) - f(x)}{a - x} = f'(y) < 0\ .$$

Because the denominator is positive, the numerator is negative and $f(a) < f(x)$. For $x \in P^+(a, \theta)$ one has that $f'(y) > 0$ and the denominator is negative, so the numerator is again negative and again $f(a) < f(x)$. Thus f has at a a strict local minimum. $\square$

The proposition says nothing about the case $f''(a) = 0$. This case can be partially resolved by generalizing the proposition to derivatives with orders > 2.

• *Convexity and concavity of functions.* Let $B := (c, d) \in \mathbb{R}^2$ be a point in the plane and ℓ be a non-vertical line, given by the equation $y = sx + b$. If the inequality $d \ge sc + b$, resp. $d > sc + b$, holds, we write $B \geq \ell$, resp. $B > \ell$, and say that B *lies above* ℓ, resp. that B *lies strictly above* ℓ. By reversing the inequalities we define that B *lies below* ℓ, resp. that B *lies strictly below* ℓ, symbolically $B \leq \ell$, resp. $B < \ell$.

Definition 10 (convex and concave) *Let $f\colon I \to \mathbb{R}$ be a function defined on an interval $I \subset \mathbb{R}$. The function is convex (on I) if for every three numbers $a < b < c$ in I the "inequality"*

$$(b,\ f(b)) \leq \kappa(a,\ f(a),\ c,\ f(c))$$

holds. If this "inequality" is strict, f is strictly convex (on I). If the opposite "inequalities" hold, we call the function f concave, resp. strictly concave, (on I).

Recall that $\kappa(a, f(a), c, f(c))$ is the secant of the graph G_f going through the points $(a, f(a))$ and $(c, f(c))$. A typical example of a strictly convex function is

$$f(x) = x^2\colon \mathbb{R} \to \mathbb{R}\ .$$

The function

$$f(x) = -x^2\colon \mathbb{R} \to \mathbb{R}$$

is then strictly concave. In general, $f\colon I \to \mathbb{R}$ is (strictly) convex $\iff -f$ is (strictly) concave. (Strict) convexity, resp. (strict) concavity, is preserved when the function is restricted to a subinterval.

We present without proof the interesting fact that convexity and concavity imply continuity, in fact even one-sided differentiability.

Theorem 11 ($\exists$ one-sided derivatives) *Every convex, resp. concave, function $f\colon I \to \mathbb{R}$ that is defined on an open interval $I \subset \mathbb{R}$ has finite one-sided derivatives*

$$f'_-,\ f'_+\colon I \to \mathbb{R}\ .$$

They are non-decreasing, resp. non-increasing.

By Proposition 5 in the last lecture, such function f is left- and right-continuous at each point in I and is therefore continuous on I. However, the (two-sided) derivative f' may not exist at some points, as the convex function $|x|$ shows.

Theorem 12 (f'', convexity and concavity) *Let $I \subset \mathbb{R}$ be an interval and let $f\colon I \to \mathbb{R}$ be a continuous function that has at each point $b \in I^0$ possibly infinite second derivative $f''(b) \in \mathbb{R}^*$. Then the following hold.*

1. *$f'' \geq 0$, resp. $f'' \leq 0$, on $I^0 \Rightarrow f$ is convex, resp. concave, on I.*
2. *$f'' > 0$, resp. $f'' < 0$, on $I^0 \Rightarrow f$ is strictly convex, resp. strictly concave, on I.*

To prove this theorem, we need the following geometric lemma, the proof of which is left as an exercise. It says that if we go from left to right and append to a non-vertical straight segment $(a, a')(b, b')$ another non-vertical straight segment $(b, b')(c, c')$ with the same or greater slope, then the common point (b, b') lies below the line going through the extreme points (a, a') and (c, c').

Lemma 13 (on slopes) *Let (a,a'), (b,b') and (c,c') be in $\mathbb{R}^2$ and $a<b<c$. Then*

$$\frac{b'-a'}{b-a}\le\frac{c'-b'}{c-b}\Rightarrow (b,\,b')\le\kappa(a,\,a',\,c,\,c')\ .$$

Furthermore, strict inequality implies strict "inequality" and both of these implications hold with opposite inequalities and "inequalities".

Proof of Theorem 12. The assumption on the existence of f'' means that there exists finite $f'\colon I^0\to\mathbb{R}$. We still have to assume the continuity of f so that it holds also in endpoints of I. Let $f''\ge 0$ on I^0, the other three cases in 1 and 2 are treated similarly. Let $a<b<c$ be any three numbers in I. By Theorem 2 there exist a $y\in(a,b)$ and a $z\in(b,c)$ such that

$$s:=\frac{f(b)-f(a)}{b-a}=f'(y)\ \text{ and }\ t:=\frac{f(c)-f(b)}{c-b}=f'(z)\ .$$

By Theorem 4, f' is non-decreasing on I^0 because f'' is non-negative. As $y<z$, the slope $s=f'(y)$ of the straight segment $(a,f(a))$ $(b,f(b))$ is at most the slope $t=f'(z)$ of the straight segment $(b,f(b))$ $(c,f(c))$. According to the previous lemma, the point $(b,f(b))$ lies below the line

$$\kappa(a,\,f(a),\,c,\,f(c))\ .$$

Thus the condition in Definition 10 holds and f is convex on I. $\Box$

• *Inflection points.* They can be defined in various ways, but for us they are the points of the graph where it passes from one side of the tangent to the other. The precise definition follows.

Definition 14 (inflection) *Let $a \in M \subset \mathbb{R}$, where a is a* TLP *of the set M, $f\colon M \to \mathbb{R}$ and ℓ be tangent to G_f at $(a, f(a))$. The point $(a, f(a))$ is called the inflection point of the graph of f, if there exists a δ such that for every $x \in P^-(a,\delta) \cap M$ and every $x' \in P^+(a,\delta) \cap M$,*

$$(x,\, f(x)) \le \ell \ \ \textit{and} \ \ (x',\, f(x')) \ge \ell \ ,$$

or the reversed "inequalities" always hold.

For example, the point $(0,0)$ is the inflection point of the graph of the function

$$f(x) = x^3\colon \mathbb{R} \to \mathbb{R} \ ,$$

because in it G_f goes from the lower to the upper side of the tangent $y = 0$ (this example falls under Theorem 16).

The following proposition provides a necessary condition for inflection: the function is differentiable at the given point (so that the tangent exists) and the second derivative at the point does not exist or is zero.

Proposition 15 (no inflection) *Let $f\colon U(a,\delta) \to \mathbb{R}$ and $\exists\, f''(a) \in \mathbb{R}^*$, but it is not zero. Then*

$$(a, f(a)) \ \textit{is not an inflection point of the graph of} \ f \ .$$

Proof. The assumption on f'' means that (after possibly decreasing δ) there exists finite $f'\colon U(a,\delta) \to \mathbb{R}$. Let $f''(a) > 0$, the case with $f''(a) < 0$ is treated similarly. Let ℓ be tangent to G_f at $(a, f(a))$, so that it has the slope $f'(a)$ and passes through the point $(a, f(a))$. By Proposition 5 there exists a $\theta \le \delta$ such that for every $x \in P^-(a,\theta)$ and every $x' \in P^+(a,\theta)$,

$$f'(x) < f'(a) \ \text{ and } \ f'(x') > f'(a)\ . \tag{1}$$

Let $x \in P^-(a,\theta)$ and $x' \in P^+(a,\theta)$ be arbitrary and let s and t be the slopes of the secants

$$\kappa(x,\ f(x),\ a,\ f(a)) \ \text{ and } \ \kappa(a,\ f(a),\ x',\ f(x'))$$

of G_f, respectively. Due to the inequalities (1) and the mean value Theorem 2 we can easily see that $s < f'(a) < t$. Hence

$$(x,\ f(x)) > \ell \ \text{ and } \ (x',\ f(x')) > \ell$$

and the condition in Definition 14 is not met. $\square$

We give without proof a sufficient condition for inflection.

Theorem 16 (inflection exists) *Let $f\colon U(a,\delta) \to \mathbb{R}$, for every $b \in U(a,\delta)$ there exists finite $f''(b)$, $f''(a) = 0$, $f'' \geq 0$ on $P^-(a,\delta)$ and $f'' \leq 0$ on $P^+(a,\delta)$ or opposite inequalities hold. Then*

$$(a,\ f(a)) \textit{ is an inflection point of the graph of } f\ .$$

• *Asymptotes of functions.* An asymptote of a function is a line, possibly vertical, to which the graph of the function gets in infinity arbitrarily close.

Definition 17 (vertical asymptotes) *Let $M \subset \mathbb{R}$, $b \in \mathbb{R}$ be a left limit point of M and $f\colon M \to \mathbb{R}$. If*

$$\lim_{x \to b^-} f(x) = \pm\infty \ ,$$

we call the line $x = b$ the left vertical asymptote of f. Right vertical asymptotes are defined similarly.

For example, the line $x = 0$ is both the left and right vertical asymptote of $f(x) = 1/x\colon \mathbb{R} \setminus \{0\} \to \mathbb{R}$. It is also the right vertical asymptote of $f(x) = \log x\colon (0, +\infty) \to \mathbb{R}$.

Definition 18 (asymptotes in infinity) *Let $M \subset \mathbb{R}$, $+\infty$ be a limit point of M, $s, b \in \mathbb{R}$ and $f\colon M \to \mathbb{R}$. If*

$$\lim_{x \to +\infty} (f(x) - sx - b) = 0 \ ,$$

we call the line $y = sx + b$ the asymptote of the function f in $+\infty$. Asymptotes in $-\infty$ are defined similarly.

Obviously, $y = sx + b$ is the asymptote of a function f in $+\infty$ iff $\lim_{x \to +\infty} f(x)/x = s$ and $\lim_{x \to +\infty}(f(x) - sx) = b$. Similarly for asymptotes in $-\infty$. For example, $y = 0 = 0x + 0$ is the asymptote of the function $f(x) = 1/x$ both in $+\infty$ and in $-\infty$.

• *Graphing functions.* To graph a function f (i.e., to make a picture of G_f), usually given by a formula, we first determine its definition domain, the set $M \subset \mathbb{R}$ maximal to inclusion such that $f\colon M \to \mathbb{R}$. Almost always it is a union of at most countably many intervals. We determine whether f is of a special form (even, odd, periodic, ...). We determine where f is continuous and where

f' exists. We find one-sided limits at the points of discontinuity of f and at the limit points of M lying outside M. We calculate one-sided derivatives, then Proposition 6 helps. Using Theorem 4 we determine maximum intervals of monotonicity. We find local and global extremes. We determine intersections of G_f with the coordinate axes and the image $f[M]$.

We determine where f'' exists and, using Theorem 12, determine maximum intervals of convexity and concavity. Using Proposition 15 and Theorem 16 we find inflection points of the graph. We determine asymptotes of the function f and draw its graph by hand or computer.

Example 1 Let

$$f(x) := \tan x = \frac{\sin x}{\cos x}\ .$$

The definition domain is

$$M = \bigcup_{n\in\mathbb{Z}} (\pi n - \pi/2,\, \pi n + \pi/2)\ .$$

It is a π-periodic function, $\sin(\pi + x) = -\sin x$ and $\cos(\pi + x) = -\cos x$. Due to the continuity of sine and cosine and due to the arithmetic of continuity, f is continuous on M. For $b(n) := \pi n + \frac{\pi}{2}$, $n \in \mathbb{Z}$, one has the limits

$$\lim_{x\to b(n)^-} f(x) = +\infty \ \text{ and } \ \lim_{x\to b(n)^+} f(x) = -\infty$$

—each line $x = b(n)$ is both the left and right vertical asymptote of f. There are no asymptotes in $-\infty$ and $+\infty$, nor the limits $\lim_{x\to\pm\infty} f(x)$ exist. Because

$$f'(x) = 1/\cos^2 x > 0 \ \text{ on } \ M\ ,$$

f increases on each interval $(b(n)-\pi, b(n))$. Because of this and the periodicity, f has no extremes. G_f intersects the y-axis only in the origin $(0,0)$ and the x-axis exactly in the points $(b(n)-\frac{\pi}{2}, 0) = (\pi n, 0)$, $n \in \mathbb{Z}$. By the above infinite limits and continuity of f (attaining intermediate values) we see that

$$f[M] = f[(b(n)-\pi,\, b(n))] = \mathbb{R}\ .$$

The second derivative is

$$f''(x) = \frac{2\sin x}{\cos^3 x}\colon M \to \mathbb{R}\ .$$

We have that $f''(x) = 0 \iff x = b(n) - \frac{\pi}{2}$, that $f'' < 0$ on $(b(n)-\pi, b(n)-\frac{\pi}{2})$ and that $f'' > 0$ on $(b(n)-\frac{\pi}{2}, b(n))$. Thus f is strictly concave on $(b(n)-\pi, b(n)-\frac{\pi}{2}]$, strictly convex on $[b(n)-\frac{\pi}{2}, b(n))$ and the inflection points are exactly

$$\big(b(n)-\tfrac{\pi}{2},\, 0\big) = (\pi(n),\, 0)\,,\ n \in \mathbb{Z}\ .$$

For sketches of G_f go to `https://www.desmos.com/calculator`.

Example 2 (after the lecture notes R. Černý a M. Pokorný, *Základy matematické analýzy pro studenty fyziky. 1*, Matfyz-Press, Praha 2020, pp. 193–194). Let

$$f(x) := \arcsin\big(2x/(1+x^2)\big)\ .$$

The definition domain is

$$M = \mathbb{R}$$

because the definition domain of arcsin is $[-1,1]$ and $2|x| \le 1 + x^2$ for every $x \in \mathbb{R}$ ($x^2 \pm 2x + 1 = (x \pm 1)^2 \ge 0$). This function is odd, i.e., $f(-x) = -f(x)$, because the functions $\sin x$, $\arcsin x$ and $\frac{2x}{1+x^2}$ are odd. According to the theorems on continuity of inverse functions, of rational functions and of composite functions, f is continuous on M. Clearly,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to+\infty} f(x) = \arcsin(0) = 0 ,$$

because $\frac{2x}{1+x^2} \to 0$ for $x \to \pm\infty$, and so $y = 0 = 0x + 0$ is the asymptote of f both in $-\infty$ and $+\infty$. There are no vertical asymptotes. The formulas for derivatives of arcsin, of composite functions and of ratios of functions yield that on the set $\{x \in \mathbb{R} \mid \frac{2x}{1+x^2} \neq \pm 1\} = \mathbb{R} \setminus \{-1, 1\}$,

$$\begin{aligned} f'(x) &= \frac{1}{\sqrt{1-(2x/(1+x^2))^2}} \cdot \frac{2\cdot(1+x^2) - 2x\cdot 2x}{(1+x^2)^2} \\ &= 2\cdot\frac{(1-x^2)/(1+x^2)^2}{|(1-x^2)/(1+x^2)|} = 2\cdot\frac{1-x^2}{|1-x^2|}\cdot\frac{1}{1+x^2} \\ &= \frac{2\cdot\operatorname{sgn}(1-x^2)}{1+x^2} . \end{aligned}$$

Obviously, $\lim_{x\to 1^\pm} f'(x) = \mp 1$. By Proposition 6 we have that $f'_\pm(1) = \mp 1$. Since f is odd, $f'_\pm(-1) = \pm 1$. Because $f' < 0$ on $(-\infty, -1)$, $f' > 0$ on $(-1, 1)$ and $f' < 0$ on $(1, +\infty)$, by Proposition 4 the function f decreases on $(-\infty, -1]$, increases on $[-1, 1]$ and decreases on $[1, +\infty)$. Also $f(x) < 0$ for $x < 0$ and $f(x) > 0$ for $x > 0$ (and $f(0) = 0$). According to these intervals of monotonicity and signs and according to the above zero limits, we see that f has at $x = -1$ the strict global minimum with the value $f(-1) = -\pi/2$, that at $x = 1$ it has symmetrically (f is odd) the strict global maximum with the value $f(1) = \pi/2$ and that f has no other local extrema. It follows that G_f intersects both coordinate axes only in $(0, 0)$ and that

$$f[M] = f[\mathbb{R}] = [-\pi/2,\, \pi/2] .$$

The second derivative is on $\mathbb{R} \setminus \{-1, 1\}$ equal to

$$f''(x) = \frac{-4x \cdot \operatorname{sgn}(1 - x^2)}{(1 + x^2)^2} \, .$$

Because $f'' < 0$ on $(-\infty, -1)$, $f'' > 0$ on $(-1, 0)$, $f'' < 0$ on $(0, 1)$, $f'' > 0$ on $(1, +\infty)$ and $f''(x) = 0 \iff x = 0$ (the second derivatives $f''(\pm 1)$ do not exist), by Theorem 12, Proposition 15 and Theorem 16, f is strictly concave on $(-\infty, -1]$, strictly convex on $[-1, 0]$, strictly concave on $[0, 1]$, strictly convex on $[1, +\infty)$ and $(0, 0)$ is the only inflection point (at the points $(-1, f(-1))$ and $(1, f(1))$ tangents do not exist). For sketches of G_f go to `https://www.desmos.com/calculator`.

THANK YOU FOR YOUR ATTENTION!