Combinatorial Algebraic Counting

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(Preliminary version, chapters with * are to be completed.)

Contents

Introduction

These lecture notes are based on the course NDMI015 *Kombinatorické počítání* (Combinatorial Counting), which I was teaching (in Czech) in the Computer Science Section of the Faculty of Mathematics and Physics of the Charles University in the summer semester of 2022. This text considerably extends the topics that I suggested and mentioned during my lectures.

I point out the following highlights. (i) Catalan numbers musing in Chapters 1, 2, 5, 6, 7, 8, 9, 14 and 15. (ii) The proof of convergence of any multivariate formal power series algebraic over convergent multivariate power series in Chapter 10. (iii) Enumeration of $(1, 2, \ldots, m+1)$-free permutations in Chapters 11, 12, 17 and 18. (iv) Enumeration of labeled k-regular graphs in Chapters 11, 12, 19 and 20.

I thank Tomáš Domes and Tomáš Hons for attending my lectures, for their questions and especially for their patience with my sometimes incoherent and incomplete presentation, for which this text hopefully makes up.

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Chapter 1

Catalan numbers and eventual linear recurrence sequences

The *Catalan numbers*

$$C_n := \frac{1}{n}\binom{2n-2}{n-1} = \frac{(2n-2)!}{n\cdot(n-1)!\cdot(n-1)!},\quad n\in\mathbb{N} := \{1,2,\dots\}\ ,$$

count very many classes of combinatorial structures ([22]) and because of this they enjoy attention of researchers in enumerative combinatorics. Their sequence begins as

$$(C_n) = (C_1,\, C_2,\, \dots) = (1,\, 1,\, 2,\, 5,\, 14,\, 42,\, 132,\, 429,\ 1430,\, 4862,\, 16796,\, \dots)\ .$$

How does one define them combinatorially? Let $|X|$ be the cardinality of a (finite) set X and, for $n \in \mathbb{N}_0 := \{0,1,\dots\}$, let $[n]$ be the set $\{1,2,\dots,n\}$, with $[0] := \emptyset$. One of the simplest self-contained combinatorial definition of C_n is that $C_1 := 1$ and for $n > 1$,

$$C_n := |\{\overline{a} \in \{-1,\, 1\}^{2n-2} \mid \forall\, j \in [2n-2]\colon\ \textstyle\sum_{i=1}^{j} a_i \ge 0 \wedge \sum_{i=1}^{2n-2} a_i = 0\}|\ .$$

So C_n counts the $(2n-2)$-tuples of $n-1$ ones and $n-1$ minus ones with nonnegative initial sums. We call such $2n$-tuples *Dyck words (with size n)* and denote their set by $\mathcal{D}_n$. Thus

$$C_n = |\mathcal{D}_{n-1}|\ .$$

For instance, there are $C_4 = 5$ Dyck words with size 3:

$$\mathcal{D}_3 = \{(1,\,-1,\,1,\,-1,\,1,\,-1),\,(1,\,1,\,-1,\,-1,\,1,\,-1),\,(1,\,-1,\,1,\,1,\,-1,\,-1),$$
$$(1,\,1,\,1,\,-1,\,-1,\,-1)\ \text{and}\ (1,\,1,\,-1,\,1,\,-1,\,-1)\}\ .$$

In the next chapter we prove in two ways the equality

$$|\mathcal{D}_{n-1}| = \frac{1}{n}\binom{2n-2}{n-1}\ ,$$

by the OGF (ordinary generating function)

$$C(x) := \sum_{n=1}^{\infty} C_n x^n \in \mathbb{C}[[x]]$$

(by $\mathbb{C}[[x]]$ we denote the ring of formal power series in variable x with complex coefficients) and then purely combinatorially.

The book [22] by R. P. Stanley contains an extensive collection of classes of combinatorial structures counted by the Catalan numbers. R. P. Stanley is the algebraic father of modern enumerative combinatorics, see [23]. The analytic father is without doubt the late P. Flajolet, the first author of [10]. Who is the combinatorial father? I do not know. Or, wait, maybe there is a combinatorial mother? M. Bousquet-Mélou, see [4, 5, 6] or her many other articles on enumeration, is the top candidate.

A class of popular sequences in enumerative combinatorics is linear recurrence sequences.

Definition 1.1 (LRS) *A linear recurrence sequence in a field K, which we abbreviate as a* LRS*, is a sequence*

$$(a_n) = (a_1,\, a_2,\, \dots) \subset K$$

such that for some $k \in \mathbb{N}_0$ (recurrence) coefficients $c_0, c_1, \dots, c_{k-1} \in K$*,*

$$\forall\, n \in \mathbb{N}\colon a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i}\ . \tag{LRS}$$

We say that k is the order of the recurrence*.*

So every term in the sequence from a_{k+1} on is a linear combination with the coefficients c_i of the k preceding terms. For $k = 0$ we define the empty sum in (LRS) as 0_K and get the *zero sequence* $(a_n) = (0_K, 0_K, \dots)$. The LRS (a_n) is uniquely determined by the coefficients c_i in (LRS) and by the *initial values* $a_1, a_2, \dots, a_k$. A well known LRS is the sequence $(F_n) \subset \mathbb{Q}$ of *Fibonacci numbers* that is given by the recurrence

$$F_{n+2} = F_{n+1} + F_n$$

and the initial values $F_1 = F_2 = 1$. Hence

$$(F_n) = (\underline{1},\, \underline{1},\, 2,\, 3,\, 5,\, 8,\, 13,\, 21,\, 34,\, 55,\, 89,\, \underline{144},\, 233,\, \dots)\ .$$

In 1964, J. H. E. Cohn proved in [8] that the underlined terms are the only square Fibonacci numbers. The book [9] is a basic monograph on LRS.

We relax the requirement in Definition 1.1.

Definition 1.2 (eventual LRS) *A sequence* $(a_n) \subset K$ *is an* eventual LRS (in K) *if for some* $n_0 \in \mathbb{N}_0$ *the recurrence* (LRS) *holds for every* $n > n_0$.

Clearly, any LRS in K is an eventual LRS in K. In Chapters 5, 6, 7 and 8 we give four proofs for the next theorem.

Theorem 1.3 ((C_n) is not an eventual LRS) *The sequence*

$$(C_n) = (1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ \dots)$$

of Catalan numbers is not an eventual LRS *in any field.*

In these proofs we follow roughly the joint article [17].

Our Definitions 1.1 and 1.2 are somewhat ambiguous as they are relative to the field K. This ambiguity will be removed in Theorem 4.1. For now we give a simple non-example of an eventual LRS.

Proposition 1.4 *The sequence of ones and zeros*

$$(a_n) := (0,\ 1,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ \dots) \subset \mathbb{Q}$$

is not an eventual LRS *in any field* $K \supset \mathbb{Q}$.

Proof. Thus (a_n) is formed by blocks of zeros with lengths $1, 2, 3, \dots$, separated by ones. Let the recurrence (LRS) hold for (a_n) for every $n > n_0$. We easily find an $m \in \mathbb{N}$ with $m > n_0$ such that $a_{m+i} = 0$ for $i = 0, 1, \dots, k-1$ and $a_{m+k} = 1$. But by the recurrence also $a_{m+k} = 0$. This is a contradiction. $\square$

LRS and eventual LRS are characterized by rationality of their generating functions. For the next theorem recall that in the the ring of formal power series $K[[x]]$ with coefficients in K the units $f(x)$ are exactly the power series with nonzero constant coefficient, i.e., such that $f(0_K) \neq 0_K$. Also recall the notation

$$[x^k]\, E,\ [x_1^{k_1} \dots x_m^{k_m}]\, E,\ \dots$$

for the coefficient of x^k or $x_1^{k_1} \dots x_m^{k_m}$ or $\dots$ in the expression E. We define the degree of the zero polynomial as $-\infty$.

Theorem 1.5 (characterization of eventual LRS) *Let* $k \in \mathbb{N}_0$, $(a_n) \subset K$ *be a sequence in a field* K *and let* $A(x) := \sum_{n \geq 1} a_n x^n \in K[[x]]$ *be its* OGF. *The following three properties of* (a_n) *and* $A(x)$ *are mutually equivalent.*

1. *The sequence* (a_n) *is an eventual* LRS *in* K *and satisfies for every* $n > n_0$ *an order* k *recurrence* (LRS).

2. *There exist polynomials* $p, q \in K[x]$ *such that* $q(x) \neq 0$, $\deg q \leq k$ *and in* $K[[x]]$ *the identity*

$$q(x)A(x) = p(x)$$

   holds.

3. *There exist polynomials $p, q \in K[x]$ such that $q(0_K) \neq 0_K$, $\deg q \leq k$ and in $K[[x]]$ the identity*
$$A(x) = \frac{p(x)}{q(x)}$$
holds.

Proof. Implication 1 $\Rightarrow$ 2. Suppose that for $k \in \mathbb{N}_0$ coefficients $c_0, \dots, c_{k-1}$ in K for every $n > n_0$ the recurrence $a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i}$ holds. Then

$$\forall\, n > n_0 \colon [x^{n+k}]\, \underbrace{(1_K - c_{k-1}x - \cdots - c_0 x^k)}_{q(x)}\, A(x) = 0_K \ ,$$

so that $q(x)A(x) = p(x)$ where $q(x) \in K[x]$ is as stated and $p(x) \in K[x]$ has degree at most $n_0 + k$.

Implication 2 $\Rightarrow$ 3. Let $q(x)A(x) = p(x)$ for some $p, q \in K[x]$ with $\deg q \leq k$ and $q(x) \neq 0_K$. Let $l \in \mathbb{N}_0$, resp. $m \in \mathbb{N}_0 \cup \{+\infty\}$, be maximum such that x^l, resp. x^m, divides $q(x)$, resp. $p(x)$. Then $q(x) = x^l q_0(x)$ and $p(x) = x^m p_0(x)$ with $q_0(0_K) \neq 0_K$ and $p_0(0_K) \neq 0_K$, and $p(x) = 0$ for $m = +\infty$. It follows that $l \leq m$. Thus
$$A(x) = \frac{x^{m-l} p_0(x)}{q_0(x)}$$
in $K[[x]]$, with $q_0(0_K) \neq 0_K$ and $\deg q_0 \leq \deg q \leq k$.

Implication 3 $\Rightarrow$ 1. Let $A(x) = p(x)/q(x)$ hold in $K[[x]]$, with $p, q \in K[x]$, $q(0_K) =: a \neq 0_K$ and $\deg q \leq k$. Then

$$(a^{-1}q(x))A(x) = a^{-1}p(x) \ .$$

Since the polynomial $a^{-1}q(x)$ has constant term 1_K and degree at most k, it is clear that (a_n) satisfies for every $n > \deg p - k$ an order k recurrence (LRS). $\Box$

For multivariate linear recurrences with constant coefficients the corresponding OGF may be much more complicated, see [6]. Since solutions $f \in K[[x]]$ of linear algebraic equations

$$a(x) \cdot f(x) + b(x) = 0$$

with polynomial coefficients $a, b \in K[x]$, $a \neq 0$, are as sequences represented by eventual LRS and not by LRS, in this approach eventual LRS and not LRS is the main actor on the sequences scene.

Chapter 2

Playing with the Catalan numbers

In the previous chapter we codefined the n-th Catalan number, besides setting $C_n := \frac{1}{n}\binom{2n-2}{n-1}$, as

$$C_n := |\mathcal{D}_{n-1}|$$

where $\mathcal{D}_n$ is the set of Dyck words with size n. At the end of this chapter we indeed show that these two definitions yield the same numbers but now we introduce another class of combinatorial structures counted by the Catalan numbers.

Definition 2.1 (rp trees) *A rooted plane tree T, or shortly an* rp tree*, with $n \in \mathbb{N}$ vertices is any set*

$$T \subset [n] \times [n]$$

such that if $(i, j) \in T$ then $i < j$ and for every vertex *$v \in [n]$ there is in T a* <u>*unique*</u> path *from the* root *$1 \in [n]$ to v. The path is a unique sequence of vertices*

$$(v_0, v_1, \ldots, v_k) \subset [n],\ k \in \mathbb{N}_0 ,$$

such that $v_0 = 1$, $v_k = v$ and $(v_{i-1}, v_i) \in T$, i.e., (v_{i-1}, v_i) is an edge *of T, for every $i = 1, 2, \ldots, k$.*

It follows that every rp tree T with n vertices has $n-1$ edges, which form the set T. We let $\mathcal{T}_n$ denote the set of rp trees with n vertices, set $\mathcal{T} := \bigcup_{n\geq 1} \mathcal{T}_n$ and define the *size* $|T|$ *of* $T \in \mathcal{T}_n$ to be $|T| := n$. For example, $\mathcal{T}_1 = \{\emptyset\}$ and

$$\begin{aligned}\mathcal{T}_4 \quad = \quad & \{\{(1, 2), (1, 3), (1, 4)\},\ \{(1, 2), (2, 3), (1, 4)\},\ \{(1, 2), (1, 3), (3, 4)\}, \\ & \{(1, 2), (2, 3), (3, 4)\}\ \text{ and }\ \{(1, 2), (2, 3), (2, 4)\}\}\ .\end{aligned}$$

Proposition 2.2 (rp trees and C_n) *For every $n \in \mathbb{N}$,*

$$|\mathcal{T}_n| = |\mathcal{D}_{n-1}| = C_n\ .$$

Proof. For $n = 1$ the claim holds trivially. We assume that $n \geq 2$ and define a bijection

$$F\colon \mathcal{T}_n \to \mathcal{D}_{n-1}\ .$$

Let $T \in \mathcal{T}_n$. We define the *go-around sequence* of vertices

$$\overline{v} = (v_0,\, v_1,\, \ldots,\, v_{2n-2}) \subset [n]$$

associated to T as follows. It starts with the root $v_0 := 1$ and if v_i with $i < 2n-2$ is already defined, $v_{i+1} := \min j$, taken over all edges $(v_i, j) \in T$ with $j \notin \{v_0, \ldots, v_i\}$ if one exists, and else $v_{i+1} := j$ for the unique edge $(j, v_i) \in T$. So we walk (globally) clockwisely around T — we imagine T as growing up from the root 1, with every edge (i, j) directed up and with the edge (i, j') lying slightly to the right of any edge (i, j) with $j < j'$ — with the start at the root 1, and since we go along every edge twice, once up and once down, we do $2(n-1) = 2n-2$ steps. We define

$$F(T) := (d_1,\, \ldots,\, d_{2n-2}) \in \{-1,\, 1\}^{2n-2}$$

by setting $d_i = 1$ if $v_{i-1} \rightsquigarrow v_i$ is a step up, and $d_i = -1$ if it is a step down. It is easy to see that $F(T) \in \mathcal{D}_{n-1}$. The map F is a bijection because it has the inverse

$$G(\overline{d}) = G(d_1,\, \ldots,\, d_{2n-2}) := T$$

which we describe now. We get the go-around sequence $\overline{v}$ from $\overline{d} \in \mathcal{D}_{n-1}$ as follows. We initialize it by setting $i := 0$, $v_i = v_0 := 1$, $j := 0$ and then for $i < 2n-2$ we do the action $i := i+1$, $[d_i = 1 \Rightarrow v_i := 1 + \max(\{v_0, \ldots, v_{i-1}\})$, $j := j+1]$ and $[d_i = -1 \Rightarrow v_i := v_{j-1},\ j := j-1]$. We define

$$G(\overline{d}) = T := \{(v_{i-1},\, v_i) \mid i \in [2n-2] \wedge d_i = 1\}\ .$$

It is not hard to see that $T \in \mathcal{T}_n$ and that G is the inverse of F. $\square$

The next proposition is crucial for obtaining an equation for the OGF $C(x)$.

Proposition 2.3 (decomposition of rp trees) *There exists a bijection*

$$F\colon \mathcal{T} \setminus \mathcal{T}_1 \to \mathcal{T} \times \mathcal{T}$$

that is size-preserving, $F(T) = (U, V) \Rightarrow |T| = |U| + |V|$.

Proof. Let T be an rp tree with $n \geq 2$ vertices. We define its *subtree* T_2 (i.e., the subtree rooted in the vertex 2) as

$$T_2 := \{(i-1,\, j-1) \mid (i,\, j) \in T \text{ and appears in a 2-}v\text{ path}\}\ .$$

Here a 2-v path is as before any sequence of vertices $(v_0, \ldots, v_k) \subset [n]$ such that $v_0 = 2$, $v_k = v$ and $(v_{i-1}, v_i) \in T$ for every $i = 1, 2, \ldots, k$ (these paths are unique too). It follows that T_2 is an rp tree with less than n vertices.

We define the *deletion of* T_2 to be

$$T - T_2 := \{(i',\, j') \mid (i,\, j) \in T \wedge j > |T_2| + 1\}$$

where $i' := 1$ if $i = 1$ and $i' := i - |T_2|$ if $i > 1$. It follows that $T - T_2$ is an rp tree with $n - |T_2|$ vertices.

Let T and U be rp trees with m and n vertices, respectively. We define the *pairing* $V = T + U$ (of T and U) by

$$V = T + U := \{(1,\, 2)\} \cup \{(i+1,\, j+1) \mid (i,\, j) \in T\} \cup \{(i',\, j') \mid (i,\, j) \in U\}\ ,$$

where $i' := 1$ if $i = 1$ and $i' := i + m$ if $i > 1$. It follows that V is an rp tree with $m + n$ vertices and that $T = V_2$ and $U = V - V_2$.

We set $F(T) := (T_2, T - T_2)$. It is a bijection because $G(T, U) := T + U$ is its inverse map. It is clear that F is size-preserving. $\square$

The interested reader may want to compare our approach to rp trees with those in [1, 3, 10].

Thus for

$$C(x) = \sum_{n=1}^{\infty} C_n x^n = \sum_{T \in \mathcal{T}} x^{|T|}$$

the previous proposition gives the equations

$$C(x) - x = C(x) \cdot C(x) \ \text{ and } \ C(x)^2 - C(x) + x = 0\ .$$

From the former equation, or directly from the bijection F, we get the recurrence

$$C_1 = 1 \ \text{ and } \ C_n = \sum_{k=1}^{n-1} C_k C_{n-k} \ \text{ for } \ n \ge 2\ ,$$

which we call the *combinatorial recurrence (for (C_n))*.

By solving the latter quadratic equation in $\mathbb{C}[[x]]$ we obtain explicit formulas for $C(x)$ and C_n:

$$C(x) = \frac{1 - \sqrt{1-4x}}{2} = \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \binom{1/2}{n} (-4x)^n$$

and so, for $n \in \mathbb{N}$,

$$C_n = \frac{1}{2} \cdot 4^n (-1)^{n+1} \binom{1/2}{n} \qquad \text{(1st binomial formula)}$$

and

$$\begin{aligned} C_n &= 4^{n-1} \cdot \frac{(1-1/2)(2-1/2)\dots(n-1-1/2)}{n!} \\ &= 2^{n-1} \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (2n-3)}{n!} \\ &= \frac{1 \cdot 3 \cdot \ldots \cdot (2n-3) \cdot 2 \cdot 4 \cdot \ldots \cdot (2n-2)}{n!(n-1)!} \\ &= \frac{1}{n} \cdot \frac{(2n-2)!}{(n-1)!(n-1)!}\ , \end{aligned}$$

hence the classical formula

$$C_n = \frac{1}{n}\binom{2n-2}{n-1} . \qquad \text{(2nd binomial formula)}$$

The ratio of two consecutive Catalan numbers therefore equals

$$\frac{C_n}{C_{n-1}} = \frac{n-1}{n} \cdot \frac{(2n-2)(2n-3)}{(n-1)^2} = \frac{4n-6}{n}$$

and we get the recurrence

$$C_1 = 1 \text{ and } C_n = \frac{4n-6}{n} \cdot C_{n-1} \text{ for } n \geq 2 .$$

We call it the P-*recurrence (for* (C_n)*).* So (C_n) is a P-recurrent sequence by Definition 11.1.

We show how to deduce the P-recurrence for C_n from the quadratic equation for $C(x)$, so that one does not need the explicit formula for C_n. We differentiate the equation and get

$$2C(x)C(x)' - C(x)' + 1 = 0, \text{ or } C(x)' = \frac{1}{1-2C(x)} .$$

We expand the fraction by $-C(x)/2 + 1/4$ and get that

$$C(x)' = \frac{-C(x)/2 + 1/4}{1/4 - x} ,$$

which is equivalent with

$$(1-4x)C(x)' + 2C(x) - 1 = 0 .$$

Thus

$$(1-4x)\sum_{n\geq 0}(n+1)C_{n+1}x^n + 2\sum_{n\geq 1} C_n x^n - 1 = 0 .$$

Equating for $n > 1$ the coefficient of x^{n-1} on the left-hand side to zero we get that

$$nC_n - 4(n-1)C_{n-1} + 2C_{n-1} = 0 ,$$

which is a rearrangement of the P-recurrence. Thus we have deduced the P-recurrence for (C_n) only from the fact that $C(x)$ satisfies a quadratic equation. This is generalized in Proposition 11.3.

We return to the combinatorial recurrence for (C_n) and deduce by means of it two properties of the Catalan numbers.

Proposition 2.4 (parity of C_n) *The n-th Catalan number C_n is odd if and only if $n = 2^m$ for some $m \in \mathbb{N}_0$.*

Proof. It clearly holds for $n = 1$. For $n > 1$ the combinatorial recurrence

$$C_n = \sum_{k=1}^{n-1} C_k C_{n-k} = C_1 C_{n-1} + \cdots + C_{n-1} C_1$$

shows that for odd n the number C_n is even, because each summand in the sum appears twice, and that for even n one has modulo 2 that $C_n \equiv C_{n/2}^2 \equiv C_{n/2}$. The claim follows by expressing n as $n = 2^k l$ for $k \in \mathbb{N}_0$ and odd $l \in \mathbb{N}$. $\square$

The second property of the Catalan numbers we obtain from the combinatorial recurrence is an exponential upper bound on C_n. An exponential lower bound is obvious: for $n \geq 3$,

$$C_n \geq C_1 C_{n-1} + C_{n-1} C_1 \geq 2C_{n-1}$$

and, by induction, $C_n \geq 2^{n-2}$ for every $n \geq 3$. To obtain inductively an exponential upper bound is more tricky and we need the next lemma for it.

Lemma 2.5 *For every $n \geq 2$,*

$$\sum_{k=1}^{n-1} \frac{1}{k^2(n-k)^2} < \frac{8}{n^2} \; .$$

Proof. For $k, n \in \mathbb{N}$ and $k < n$ we have that

$$\frac{1}{k(n-k)} = \frac{1}{n}\left(\frac{1}{k} + \frac{1}{n-k}\right) .$$

Thus the stated sum equals

$$\frac{2}{n^2}\sum_{k=1}^{n-1}\frac{1}{k^2} + \frac{2}{n^2}\sum_{k=1}^{n-1}\frac{1}{k(n-k)} = \frac{2}{n^2}\sum_{k=1}^{n-1}\frac{1}{k^2} + \frac{4}{n^3}\sum_{k=1}^{n-1}\frac{1}{k} \; .$$

For $k \geq 2$ we get from $1/k^2 < 1/k(k-1) = 1/(k-1) - 1/k$ that the last but one sum is less than 2. The last sum is trivially less than n. This gives the stated bound. $\square$

Proposition 2.6 (an exponential upper bound on C_n) *For every $n \in \mathbb{N}$,*

$$C_n \leq \frac{8^{n-1}}{n^2} \; .$$

Proof. By induction on n. For $n = 1$, $C_1 = 1 \leq 8^0/1^2 = 1$. For $n \geq 2$ we get by the combinatorial recurrence, induction and the previous lemma that

$$C_n = \sum_{k=1}^{n-1} C_k C_{n-k} \leq \sum_{k=1}^{n-1} \frac{8^{k-1}}{k^2} \cdot \frac{8^{n-k-1}}{(n-k)^2} < \frac{8^{n-2} \cdot 8}{n^2} = \frac{8^{n-1}}{n^2} \; ,$$

so that the bound holds again. $\square$

Thus we have proved for the OGF $C(x) = \sum_{n=1}^{\infty} C_n x^n$ that

$$C(x)^2 - C(x) + x = 0 \rightsquigarrow C_1 = 1 \wedge C_n = \sum_{k=1}^{n-1} C_k C_{n-k} \rightsquigarrow C_n \leq 8^{n-1}/n^2 \ .$$

Chapter 10 deals with a far reaching generalization of this result.

Precise asymptotics for $n \to \infty$ of the Catalan numbers is easily obtained from the formula $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ and the Stirling formula $n! = (1+o(1)) \cdot \sqrt{2\pi n} \cdot (n/e)^n$: for some constant $c > 0$,

$$C_n = (c + o(1)) \cdot n^{-3/2} \cdot 4^n$$

We obtain in a self-contained manner from the two above binomial formulas an estimate that is almost as good. In fact, it is in a way better because it is completely explicit and has no undetermined constants and thresholds (which are implied by the $o(1)$ term of the precise asymptotics). We use this estimate in the first proof of Theorem 1.3 in Chapter 5.

Proposition 2.7 (elementary asymptotics for C_n)

$$\forall\, n \in \mathbb{N}\colon \frac{4^n}{8n^2} < C_n \leq \frac{4^n}{4n} \ .$$

Proof. 1st proof. For every $n \in \mathbb{N}$,

$$\frac{1}{4n^2} < \left| \binom{1/2}{n} \right| = \frac{\frac{1}{2}\prod_{i=1}^{n-1}(i - 1/2)}{n!} \leq \frac{1}{2n} \ ,$$

with the empty product defined as 1, and from the 1st binomial formula above we get the estimate.

2nd proof. For every $n \in \mathbb{N}$,

$$\binom{2n-2}{n-1} \leq 4^{n-1} = (1+1)^{2n-2} = \sum_{i=0}^{2n-2} \binom{2n-2}{i} < 2n \binom{2n-2}{n-1}$$

because $\binom{2n-2}{n-1}$ is the largest of the $2n-1$ binomial coefficients in the sum. Hence from the 2nd binomial formula above we get the same estimate that for every $n \in \mathbb{N}$,

$$\frac{4^n}{8n^2} < C_n = \frac{1}{n}\binom{2n-2}{n-1} \leq \frac{4^n}{4n} \ .$$

$\square$

It is remarkable that two rather different arguments produce identical estimates.

Last topic in this chapter is a combinatorial derivation of the formula $C_n = \frac{1}{n}\binom{2n-2}{n-1}$. Recall that $\mathcal{D}_n$ is the set of $2n$-tuples of n ones and n minus ones with nonnegative initial sums and that

$$C_n := |\mathcal{D}_{n-1}|\ .$$

We define $\mathcal{E}_n$ and $\mathcal{F}_n$ to be, respectively, the set of $\binom{2n}{n}$ $2n$-tuples of of n ones and n minus ones and the set of $\binom{2n}{n+1} = \binom{2n}{n-1}$ $2n$-tuples of of $n+1$ ones and $n-1$ minus ones. We show that there exists a bijection

$$F\colon \mathcal{E}_n \setminus \mathcal{D}_n \to \mathcal{F}_n\ .$$

Then, since $\mathcal{D}_n \subset \mathcal{E}_n$ and $|\mathcal{F}_{n-1}| = |\mathcal{E}_{n-1}| - |\mathcal{D}_{n-1}|$,

$$C_n = |\mathcal{D}_{n-1}| = |\mathcal{E}_{n-1}| - |\mathcal{F}_{n-1}| = \binom{2n-2}{n-1} - \binom{2n-2}{n-2} = \frac{1}{n}\binom{2n-2}{n-1}\ .$$

If $u = (u_1, \dots, u_{2n}) \in \mathcal{E}_n \setminus \mathcal{D}_n$ and $m \in [2n]$ is minimum such that $u_1 + \cdots + u_m = -1$, we define

$$F(u) := (u_1',\, \dots,\, u_m',\, u_{m+1},\, \dots,\, u_{2n})$$

where $u_i' := -u_i$. It is easy to see that $F(u) \in \mathcal{F}_n$ and thus $F\colon \mathcal{E}_n \setminus \mathcal{D}_n \to \mathcal{F}_n$. We define similarly the map

$$G\colon \mathcal{F}_n \to \mathcal{E}_n \setminus \mathcal{D}_n$$

by exchanging ones and minus ones in the shortest initial segment with sum $+1$. It is easy to check that

$$F(G) = \mathrm{id}_{\mathcal{F}_n} \ \text{ and } \ G(F) = \mathrm{id}_{\mathcal{E}_n \setminus \mathcal{D}_n}\ ,$$

so that F and G are bijections.

Chapter 3

*Three results on power sums

For the first proof of Theorem 1.3 in Chapter 5 we need to establish an explicit formula for an eventual LRS and, in order to contradict the growth of (C_n), to show that the leading term of this formula does not asymptotically vanish. These two results are obtained in the present chapter. The explicit formula for any eventual LRS generalizes well known *Binet's formula* for the Fibonacci numbers, named after J. P. M. Binet (1786–1856):

$$\forall\, n \in \mathbb{N}\colon F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) .$$

Theorem 3.1 (generalized Binet's formula) *If* $(a_n) \subset \mathbb{C}$ *is an* eventual LRS in $\mathbb{C}$ *then*

$$\forall\, n > n_0\colon a_n = \sum_{j=1}^{l} p_j(n)\alpha_j^n \ ,$$

where $l \in \mathbb{N}_0$, for $l = 0$ the sum is defined as 0, $p_j \in \mathbb{C}[x]$ are nonzero polynomials and the numbers $\alpha_j \in \mathbb{C}$ are nonzero and mutually distinct.

More generally, for any field K we call the formal expression

$$s(x) := \sum_{j=1}^{l} p_j(x)\alpha_j^x \ ,$$

where $l \in \mathbb{N}_0$, $p_j \in K[x]$ are nonzero polynomials and $\alpha_j \in K \setminus \{0_K\}$ are distinct *roots*, a *power sum (over K)*. For $l = 0$ we get the *empty power sum*. Every power sum defines in a natural way a function $s\colon \mathbb{N} \to K$, even $s\colon \mathbb{Z} \to K$. For the empty power sum it is the zero sequence.

To obtain Theorem 3.1 we prove by generating functions a more precise result. In order that we can work in $K[[x]]$ easily, we switch for this result to sequences $(a_n)_{n\ge 0} \subset K$ indexed by $\mathbb{N}_0$. Let $k \in \mathbb{N}_0$ and $c_0, \dots, c_{k-1}$ be in a field K. We define the set of such sequences

$$U(c_0,\,\dots,\,c_{k-1}) := \{(a_n)_{n\ge 0} \subset K \mid \forall\, n \in \mathbb{N}_0\colon a_{n+k} = \textstyle\sum_{i=0}^{k-1} c_i a_{n+i}\}\ ,$$

where for $k = 0$ we define this set as $\{(0_K, 0_K, \dots)_{n\ge 0}\}$, i.e., as consisting only of the zero sequence. Thus $U(c_0,\,\dots,\,c_{k-1})$ are just the LRS (a_n) satisfying the recurrence (LRS) and extended by the term a_0. Let

$$q(x) = q_{c_0,\dots,c_{k-1}}(x) := 1_K - c_{k-1}x - \dots - c_0 x^k \in K[x]\ ,$$

where for $k = 0$ we set $q(x) := 1_K$. We call this polynomial the *characteristic polynomial (of the recurrence)*. If K is algebraically closed and $c_0 \ne 0_K$, then $\deg q = k$ and we have the factorization

$$q(x) = \prod_{i=1}^{l} (1_K - \alpha_i x)^{m_i}\ ,$$

where $l \in \mathbb{N}_0$, $\alpha_i \in K$ are nonzero and distinct and $m_i \in \mathbb{N}$ sum up to k. For $k = 0$ we set $l := 0$ and define the factorization as $q(x) = 1_K$. We consider the second set of sequences

$$\begin{aligned} V(c_0,\,\dots,\,c_{k-1}) \quad := \quad & \{(a_n)_{n\ge 0} \subset K \mid \forall\, n \in \mathbb{N}_0\colon a_n = \textstyle\sum_{i=1}^{l} p_i(n)\alpha_i^n\ , \\ & \text{for some } p_i \in K[x] \text{ with degrees } \deg p_i < m_i\}\ , \end{aligned}$$

where for $k = 0$ we again define this set as $\{(0_K,\, 0_K,\, \dots)_{n\ge 0}\}$. We show that these two sets of sequences in fact coincide. The elegant proof method using vector spaces is due to [21, Theorem?].

Theorem 3.2 (concrete power sums) *Let K be an algebraically closed field with characteristic 0, $k \in \mathbb{N}_0$, $c_0, \dots, c_{k-1}$ be in K but with $c_0 \ne 0_K$ and let the characteristic polynomial $q(x) \in K[x]$ be factorized as above. Then*

$$U(c_0,\,\dots,\,c_{k-1}) = V(c_0,\,\dots,\,c_{k-1})\ .$$

Proof. It is clear that both $U = U(c_0, \dots, c_{k-1})$ and $V = V(c_0, \dots, c_{k-1})$ are subspaces of the K-vector space of sequences $(a_n)_{n\ge 0} \subset K$ and that they have the same dimension k. We will be done if we show that $V \subset U$ (then also $U \subset V$, which is the inclusion we interested in). We accomplish it by means of generating functions.

For $m \in \mathbb{N}$ and $\alpha \in K$ we have the expansion

$$(1_K - \alpha x)^{-m} = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1}_K x^n\ .$$

For $n \in \mathbb{N}_0$ the polynomial

$$p_n(x) := \binom{x+n}{n}_K = \frac{(x+n_K)(x+n_K-1_K)\dots(x+1_K)}{n!_K} \in K[x]$$

has degree n and thus $\{p_n(x) \mid n = 0, 1, \ldots, m-1\}$ is a basis of the K-vector space of polynomials in $K[x]$ with degrees less than m. Here one runs in difficulties for K with characteristic p because then $n!_K = 0_K$ for every $n > n_0$. Thus

$$B := \{(1_K - \alpha_i x)^{-n_i} \mid i = 1,\, 2,\, \ldots,\, l \text{ and } n_i = 1,\, 2,\, \ldots,\, m_i\}$$

is a basis of the vector subspace $V_{\mathrm{OGF}} \subset K[[x]]$ where we represent the sequences in V by their OGF. By bringing the rational functions in B on the common denominator $q(x)$, we see that every $\sum_{n\geq 0} a_n x^n \in V_{\mathrm{OGF}}$ has the expression

$$\sum_{n\geq 0} a_n x^n = \frac{p(x)}{q(x)}$$

for some polynomial $p(x) \in K[x]$ with degree $\deg p < \deg q = k$. Hence

$$[x^{n+k}]\, q(x) \sum_{n\geq 0} a_n x^n = [x^{n+k}]\, p(x) = 0_K$$

for every $n \in \mathbb{N}_0$, which means that $a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i}$ for every $n \in \mathbb{N}_0$. Thus $\sum_{n\geq 0} a_n x^n \in U_{\mathrm{OGF}}$ and indeed $V_{\mathrm{OGF}} \subset U_{\mathrm{OGF}}$. □

Proof of Theorem 3.1. We assume that $(a_n) \subset \mathbb{C}$ is a sequence that satisfies for some coefficients $c_0, \ldots, c_{k-1} \in \mathbb{C}$, $k \in \mathbb{N}_0$, for every $n > n_0$ the recurrence $a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i}$. If $k = 0$ or if all $c_i = 0$ then (a_n) is expressed for $n > n_0$ by the empty power sum. So we assume that $k \geq 1$ and take the minimum $l \in \{0, 1, \ldots, k-1\}$ with $c_l \neq 0$. We define the sequence $(b_n)_{n\geq 0} \subset \mathbb{C}$ as $b_n := a_n$ for $n > n_0$, and for $n = n_0, n_0 - 1, \ldots, 1, 0$ by the backward recurrence, i.e., by

$$b_n = c_l^{-1} b_{n+k-l} + \sum_{i=l+1}^{k-1} (-c_i/c_l) b_{n+i-l}\ .$$

If we denote $d_0 := c_l \neq 0$, $d_1 := c_{l+1}$, $\ldots$, $d_{k-l-1} := c_{k-1}$, then the definition of $(b_n)_{n\geq 0}$ implies that

$$(b_n)_{n\geq 0} \in U(d_0,\, \ldots,\, d_{k-l-1})\ .$$

Therefore by Theorem 3.2 also

$$(b_n)_{n\geq 0} \in V(d_0,\, \ldots,\, d_{k-l-1})\ .$$

Thus b_n is expressed by a power sum for every $n \in \mathbb{N}_0$, and a_n is expressed by the same power sum for every $n > n_0$. □

The previous theorem is a staple in the theory of LRS but the next one is much less known, even though it is clearly basic in the theory of power sums. We could not find it clearly stated and proven anywhere and so we stated and proved it in [17].

Theorem 3.3 (the leading term $\not\to 0$) *Consider any power sum over* $\mathbb{C}$

$$s(n) := \sum_{j=1}^{l} \beta_j \alpha_j^n$$

with $l \in \mathbb{N}$, $\alpha_j, \beta_j \in \mathbb{C} \setminus \{0\}$, $|\alpha_j| = 1$ *for every* $j = 1, 2, \ldots, l$ *and with the numbers* α_j *all distinct. Then*

$$\limsup_{n\to\infty} |s(n)| > 0 \ .$$

In other words, there is a real constant $d > 0$ *such that*

$$|s(n)| > d$$

holds for infinitely many $n \in \mathbb{N}$.

Proof. We suppose for the contrary that $\lim_{n\to\infty} s(n) = 0$ and deduce the contradiction that $\beta_1 = \cdots = \beta_l = 0$. Let an $\varepsilon > 0$ be given. We fix an $n \in \mathbb{N}$ such that

$$|s(n+i)| < \varepsilon,\ i = 1,\, 2,\, \ldots,\, l\ ,$$

and consider the linear system

$$\sum_{j=1}^{l} \beta_j \alpha_j^{n+i} = s(n+i),\ i = 1,\, 2,\, \ldots,\, l\ ,$$

in the "unknowns" β_j. We solve it by means of Cramer's formula:

$$\beta_j = \frac{\det A(j,\, n)}{\det \left(\alpha_j^{n+i}\right)_{1\le i,j\le l}}\ ,$$

where the matrix $A(j, n)$ in the numerator arises from that in the denominator by replacing the j-th column by the column $(s(n+1), \ldots, s(n+l))^T$. After recalling the definition of determinant and the formula for Vandermonde determinants we see that

$$|\det A(j,\, n)| < l! \cdot \varepsilon \ \text{ and } \ \left|\det \left(\alpha_j^{n+i}\right)_{1\le i,j\le l}\right| = \prod_{1\le i<j\le l} |\alpha_i - \alpha_j| =: c > 0\ .$$

Thus $|\beta_j| < l! \cdot \varepsilon / c$ for every $j = 1, 2, \ldots, l$. Since ε may be arbitrarily small, all $\beta_j = 0$ but this is a contradiction. $\square$

In the heading of the theorem we should speak more precisely of the "oscillating factor in the leading term" because every nonempty power sum over $\mathbb{C}$ can be written as a sum of its leading term and its error term as

$$\sum_{j=1}^{l} p_j(n)\alpha_j^n = n^k c^n \underbrace{\sum_{j=1}^{m} \beta_j \gamma_j^n}_{s(n)} + O(n^{k-1}c^n) \tag{L+E}$$

where $l, m \in \mathbb{N}$, $k \in \mathbb{N}_0$, $\alpha_j \in \mathbb{C}$ are nonzero and distinct roots of the power sum, $c := \max_j |\alpha_j| > 0$, $p_j \in \mathbb{C}[x]$ are nonzero polynomials, $\beta_j \in \mathbb{C}$ are nonzero numbers and $\gamma_j \in \mathbb{C}$ are distinct numbers that all lie on the complex unit circle. From Theorems 3.1 and 3.3 we get at once the next fundamental estimate for every eventual LRS which we state here as a mere corollary.

Corollary 3.4 (bounding any eventual LRS) *Suppose that*

$$(a_n) \subset \mathbb{C}$$

is an eventual LRS in $\mathbb{C}$. *Then either $a_n = 0$ for every $n > n_0$ or there exist real constants $c_1 > 0$, $c_2 > c_3 > 0$ and a number $k \in \mathbb{N}_0$ such that*

$$c_3 n^k c_1^n < |a_n| < c_2 n^k c_1^n$$

holds for infinitely many $n \in \mathbb{N}$.

Proof. If $a_n = 0$ for $n > n_0$ then the claim holds trivially. Else we express a_n by means of Theorem 3.1 for $n > n_0$ as values of a nonempty power sum. In its decomposition (L+E) we have by Theorem 3.3 that $|s(n)| > d > 0$ for infinitely many $n \in \mathbb{N}$ and a constant d. Trivially,

$$|s(n)| \le |\beta_1| + \cdots + |\beta_m|$$

for every $n \in \mathbb{N}$. The stated estimate follows. □

Problem 3.5 (T. Skolem, 1930s) *Is there an algorithm*

$$\mathcal{A}\colon \bigcup_{k=0}^{\infty} \mathbb{Z}^{2k} \to \{yes, no\}$$

that decides vanishing of integral LRS*? In more words, for every $2k$-tuple*

$$a_1,\ \ldots,\ a_k,\ c_0,\ \ldots,\ c_{k-1}$$

of integers,

$$\mathcal{A}(a_1,\ \ldots,\ a_k,\ c_0,\ \ldots,\ c_{k-1}) = yes$$

if and only if the sequence $(a_n) \subset \mathbb{Z}$, given by the initial values $a_1, \ldots, a_k$ and for $n \in \mathbb{N}$ by the recurrence $a_{n+k} = \sum_{i=0}^{k-1} c_i a_{n+i}$, is ever zero, i.e., $a_n = 0$ for some $n \in \mathbb{N}$.

Chapter 4

*Two results on LRS

Theorem 4.1 (defining eventual LRS) *Suppose that $K \subset L$ is an extension of fields and that*

$(a_n) \subset K$ *is an* eventual LRS *in L. Then (a_n) is an* eventual LRS *in K.*

In more words, if $(a_n) \subset K$ and satisfies for every $n > n_0$ an order k recurrence (LRS) *with coefficients in L, then it satisfies for every $n > n_0$ an an order k recurrence* (LRS) *with coefficients in K and order at most k.*

Proof.

□

Theorem 4.2 (the Fatou lemma)

Proof.

□

Chapter 5

Proof 1 that Catalan numbers are not an eventual LRS

We give our first and **asymptotic** proof of Theorem 1.3 that the sequence (C_n) of Catalan numbers is not an eventual linear recurrence sequence. The proof follows immediately from Proposition 2.7 and Corollary 3.4 (and Theorem 4.1, to be precise). It is clear that the former elementary asymptotics of C_n is incompatible with any of the three cases $0 < c < 4$, $c = 4$ and $c > 4$ of the latter estimate. The crucial point is that the exponent k in the factor n^k in Corollary 3.4 is a *nonnegative* integer, whereas the precise asymptotics of C_n involves the power $n^{-3/2}$. $\square$

Chapter 6

Proof 2 that Catalan numbers are not an eventual LRS

Here is our second proof, by means of **generating functions**, of Theorem 1.3 that the sequence (C_n) of Catalan numbers is not an eventual linear recurrence sequence. In Chapter 2 we derived a formula for the OGF $C(x)$ of Catalan numbers,

$$C(x) = \sum_{n \geq 1} C_n x^n = \frac{1 - \sqrt{1-4x}}{2} \ .$$

Suppose for the contrary that (C_n) is an eventual LRS in $K \supset \mathbb{Q}$. By Theorem 1.5, $q(x)C(x) = p(x)$ for some polynomials $p, q \in K[x]$ with $q \neq 0_K$. Thus

$$q(x)(1 - \sqrt{1-4x})/2 = p(x) \ \text{ and } \ (1-4x)q(x)^2 = (1 - 2p(x))^2 \ .$$

This is a contradiction: the left-hand side is a nonzero polynomial with an odd degree, but the right-hand side has even degree. $\square$

Chapter 7

Proof 3 that Catalan numbers are not an eventual LRS

This is our third and **number-theoretic** proof of Theorem 1.3 that the sequence (C_n) of Catalan numbers is not an eventual linear recurrence sequence. This proof uses the fact that odd values of C_n are increasingly isolated, see Proposition 2.4, and its idea is the same as in Proposition 1.4. Using Theorem 4.1 we suppose for contradiction that (C_n) is an eventual LRS in $\mathbb{Q}$. So there exist fractions $c_0, \dots, c_k = 1$, $k \in \mathbb{N}_0$, such that for every $n > n_0$,

$$c_k C_{n+k} + \cdots + c_1 C_{n+1} + c_0 C_n = 0 \; .$$

Multiplying by a natural number we achieve that the c_i are altogether coprime integers. In particular, they are not all zero and some c_j for $j \in \{0, 1, \dots, k\}$ is odd. Using Proposition 2.4 we easily take an $n > n_0$ such that C_{n+j} is odd but all other C_{n+i} for $i \in \{0, 1, \dots, k\} \setminus \{j\}$ are even. But for this n the above displayed equality is impossible by parity: it has odd left-hand side but even right-hand side. □

Chapter 8

Proof 4 that Catalan numbers are not an eventual LRS

Our fourth, last and **polynomial** proof of Theorem 1.3 that the sequence (C_n) of Catalan numbers is not an eventual linear recurrence sequence in $K \supset \mathbb{Q}$ implements the idea that by substituting the explicit formula $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ in the recurrence one gets a nonzero polynomial with infinitely many roots, which is impossible. Suppose that

$$\forall\, n > n_0\colon \sum_{i=0}^{k} d_i C_{n+i} = 0$$

for some $k \in \mathbb{N}_0$ coefficients $d_i \in K$ with $d_k = 1$. For $l \in \mathbb{N}_0$ we define

$$(x)_l := x(x+1)\dots(x+l-1) \in \mathbb{Z}[x], \quad (x)_0 := 1\ .$$

Since

$$C_n = \frac{1}{n} \cdot \frac{(2n-2)!}{(n-1)!^2}\ ,$$

by multiplying the above displayed recurrence by

$$\frac{(n)_{k+1} \cdot (n+k-1)!^2}{(2n-2)!}$$

we get that indeed

$$\forall\, n > n_0\colon p(n) := \sum_{i=0}^{k} d_i \cdot \frac{(n)_{k+1}}{n+i} \cdot (n+k-1)_{k-i}^2 \cdot (2n-2)_{2i} = 0\ .$$

These are values of a polynomial $p(x) \in K[x]$ with degree at most $3k$. It is not the zero polynomial because

$$p(-k) = \underbrace{d_k}_{=1} (-k)_k \cdot 1^2 \cdot (-2k-2)_{2k} \neq 0 \ .$$

At the same time $p(x)$ has infinitely many zeros $n_0 + 1, n_0 + 2, \ldots,$ which is a contradiction. $\square$

Chapter 9

*Remarks on effectiveness and efficiency of these proofs

I am a mathematician (like the author of [24] was) but I also work in the Computer Science Section of . . . and so I should not be content with just proving that the sequence (C_n) of Catalan numbers violates any rational recurrence (LRS) infinitely often. I should seek an (efficient) algorithm that actually produces these violations. Thus we consider an effective version of Theorem 1.3.

Theorem 9.1 (effective Theorem 1.3) *There exists an algorithm*

$$\mathcal{A}\colon \bigcup_{k=0}^{\infty} \mathbb{Q}^k \times \mathbb{N} \to \mathbb{N}$$

such that if $n := \mathcal{A}(c_0, \dots, c_{k-1}, n_0)$ *then*

$$n > n_0 \wedge C_{n+k} \neq \sum_{i=0}^{k-1} c_i C_{n+i} \ .$$

In this chapter we show that each of the four proofs in the four preceding chapters can be made effective so that it yields an algorithm as required in the theorem.

We begin with the simple effective solution provided by **our proof no. 3** from number theory. Now for the input $c_0, \dots, c_{k-1} \in \mathbb{Q}$ and $n_0 \in \mathbb{N}$ the algorithm $\mathcal{A}$ easily finds an $m \in \mathbb{N}$ and a $j \in \{0, 1, \dots, k\}$ such that for $i = 0, 1, \dots, k$ every $mc_i \in \mathbb{Z}$ (where $c_k := 1$) and that mc_j is odd. Then $\mathcal{A}$ easily computes the minimum $l \in \mathbb{N}$ such that $2^l - k > n_0$. Finally, $\mathcal{A}$ outputs $n := 2^l - j$. By our proof no. 3, the output n has the stated properties. $\mathcal{A}$ can clearly compute efficiently, in time (i.e., number of steps) polynomial in the size of the input, which is

$$\|n_0\| + \sum_{i=0}^{k-1} \|c_i\| \ ,$$

where $\|n\| := \log(3 + |n|)$ for numbers $n \in \mathbb{Z}$ and $\|a/b\| := \max(\|a\|, \|b\|)$ for fractions $a/b \in \mathbb{Q}$ in lowest terms. However, note that in order to compute efficiently, $\mathcal{A}$ cannot find the common denominator m of the fractions c_i naively by factorizing their denominators in primes; at present nobody knows how to do it efficiently (and deterministicly), see for example the interesting text [15] by M. Hittmeir. In fact, a simpler and more natural procedure is to compute for $i = 0, 1, \ldots, k$ the 2-adic orders $r_i := \mathrm{ord}_2(c_i) \in \mathbb{Z} \cup \{+\infty\}$, by dividing numerators and denominators of the fractions c_i repeatedly by 2, and then to take any j with the minimum value of r_j.

We turn to **our proof no. 2** which uses generating functions. In order to make it effective we have to resolve the following problem. Let $a_1, \ldots, a_{2k-1}$, $k \in \mathbb{N}$ and $a_{2k-1} \neq 0$, be some fractions. Then we have in $\mathbb{Q}[[x]]$ the equality

$$\sqrt{1 + a_1 x + \cdots + a_{2k-1} x^{2k-1}} = \sum_{n=0}^{\infty} b_n x^n \ ,$$

in which the right-hand side is not a polynomial, i.e., $b_n \neq 0$ for infinitely many $n \in \mathbb{N}$. This is exactly what comparison of degrees on both sides after squaring gives. But now, given effectively the coefficients a_i, we need an effective (i.e., explicit) upper bound on the growth of nonzero coefficients b_n. Fortunately, we already know the method to get it — we applied it to derive the P-recurrence for C_n from the quadratic equation for $C(x)$ in Chapter 2.

Proposition 9.2 (on zero coefficients) *Suppose that K is a field with characteristic 0, $k \in \mathbb{N}$, $a_1, \ldots, a_{2k-1} \in K$ with $a_{2k-1} \neq 0$ and*

$$A(x) := \sqrt{1 + a_1 x + \cdots + a_{2k-1} x^{2k-1}} =: \sum_{n=0}^{\infty} b_n x^n \in K[[x]] \ .$$

Then for every $n \in \mathbb{N}_0$ there is an $i \in \{0, 1, \ldots, 2k-2\}$ such that $b_{n+i} \neq 0$.

Proof. Setting $a_0 = 1$ and equating for $n \geq 2k-2$ the coefficient of x^n on the left-hand side in

$$2 \textstyle\sum_{i=0}^{2k-1} a_i x^i \cdot A'(x) - \sum_{i=1}^{2k-1} i a_i x^{i-1} \cdot A(x) = 0$$

to zero we get the relation

$$(2n+2) b_{n+1} + \textstyle\sum_{i=1}^{2k-1} a_i (2n + 2 - 3i) b_{n+1-i} = 0 \ .$$

It shows that if $2k-1$ consecutive coefficients b_m, b_{m+1}, $\ldots$, b_{m+2k-2} vanish, then $b_n = 0$ for every $n \geq m$. But as we noted above, this is impossible $\square$

Thus if $q \in \mathbb{Q}[x]$ is any polynomial with constant term 1 and degree at most $k \in \mathbb{N}_0$ then in the formal power series

$$q(x)\sqrt{1-4x} = \sqrt{(1-4x)q(x)^2}$$

among every $2k+1$ consecutive coefficients there is a nonzero one. Hence in the formal power series

$$q(x)C(x) = q(x)/2 - q(x)\sqrt{1-4x}/2$$

among every $2k+1$ consecutive coefficients of x^n, starting from x^{k+1}, there is a nonzero one. Thus we have the next strengthening of Theorem 1.3.

Corollary 9.3 ((C_n) is often not an eventual LRS) *Let $k \in \mathbb{N}_0$ and let c_0, $\dots$, c_{k-1} be in $\mathbb{Q}$. Then for every $n \ge k+1$ there is a $j \in \{0, 1, \dots, 2k\}$ such that*

$$C_{n+k+j} \ne \sum_{i=0}^{k-1} c_i C_{n+i+j} \ ,$$

where (C_n) is the sequence of Catalan numbers.

The working of the algorithm $\mathcal{A}$ is trivial. For the input $c_0, \dots, c_{k-1} \in \mathbb{Q}$ and $n_0 \in \mathbb{N}$ it just checks validity of the recurrence for $2k+1$ consecutive n starting from $\max(k+1, n_0+1)$ and selects one for which the recurrence is violated.

To make effective **our polynomial proof no. 4** is easy. One only needs to exhibit some effective bound on sizes of zeros of any nonzero polynomial $p(x)$ in terms of its degree and sizes of its coefficients. So let

$$p(x) := \sum_{i=0}^{n} a_i x^i$$

where $n \in \mathbb{N}$, $a_i \in \mathbb{C}$ and $a_n \ne 0$. For $z \in \mathbb{C}$ with $|z| \ge \frac{2n}{|a_n|} \max_{0 \le i \le n} |a_i| \ge 2$ we have that

$$|p(z)| \ge |z|^n \left(|a_n| - \sum_{i=0}^{n-1} \frac{|a_i|}{|z|^{n-i}} \right) \ge |z|^n \left(|a_n| - \sum_{i=0}^{n-1} \frac{|a_n|}{2n} \right) = \frac{|z|^n \cdot |a_n|}{2} > 0 \ .$$

Thus we have the next bound.

Proposition 9.4 (a bound on zeros) *If $p(x) = \sum_{i=0}^{n} a_i x^i \in \mathbb{C}[x]$ has degree $\deg p = n \in \mathbb{N}$ then its every complex zero has modulus less than*

$$\frac{2n \cdot \max_{0 \le i \le n} |a_i|}{|a_n|} \ .$$

The polynomial in proof no. 4 is

$$p(x) = \sum_{i=0}^{n} a_i x^i := \sum_{i=0}^{k} d_i \cdot \frac{(x)_{k+1}}{x+i} \cdot (x+k-1)_{k-i}^2 \cdot (2x-2)_{2i} \ ,$$

where $d_0, \ldots, d_{k-1} \in \mathbb{Q}$ are minus recurrence coefficients and $d_k = 1$, and has degree at most $3k$ and is nonzero. Let $d := \max_{0 \le i \le k} |d_i|$ and let $m \in \mathbb{N}$ be a common denominator of $d_0, \ldots, d_k$. Then it is not hard to see that

$$\max_{0 \le i \le n} |a_i| \le (k+1)d \cdot (k+1)! \cdot (2k)!^2 \cdot (2k+3)! =: D \text{ and } |a_n| \ge \frac{1}{m} \, .$$

In view of Proposition 9.4 and these bounds, the algorithm $\mathcal{A}$ only needs to output the value

$$n := \max(6k\lceil D \rceil m, \, n_0 + 1) \, .$$

It clearly has the stated property.

We conclude this chapter with effectivisation of **our asymptotic proof no. 1**.

Chapter 10

*A result on convergence of formal power series

Theorem 10.1 *Let $P = P(X, y)$ be a nonzero polynomial in $K\{X\}[y]$ where $X = x_1, \ldots, x_m$ denotes a tuple of variables x_i and let $f = f(X)$ be a formal power series in $K[[X]]$ such that*

$$P(X,\, f) = P(X,\, f(X)) = 0$$

holds as an identity in $K[[X]]$. Then $f \in K\{X\}$, which is to say that there exist real constants $c, d > 1$ such that for every $(n_1, \ldots, n_m) \in \mathbb{N}_0^m$,

$$|\,[x_1^{n_1} \ldots x_m^{n_m}]\, f(X)\,| < dc^{n_1+\cdots+n_m}\ .$$

Chapter 11

*D-finiteness and P-recursiveness

Definition 11.1 (P-recurrent sequences) *We say that a sequence $(a_n) \subset K$, where K is a field, is* P-recurrent *if there exist $k \in \mathbb{N}_0$ rational functions $c_0(x), \ldots, c_{k-1}(x)$ in $K(x)$ such that*

$$a_{n+k} = \sum_{i=0}^{k-1} c_i(n) a_{n+i} \tag{P-rec}$$

holds for every $n > n_0$.

So, for example, (C_n) is a P-recurrent sequence. We cannot take $n \in \mathbb{N}$ as for LRS because of the possible zeros of the denominators in the coefficients $c_i(x)$. Or maybe we can because an equivalent definition of P-recurrence is that

$$b_k(n) a_{n+k} = \sum_{i=0}^{k-1} b_i(n) a_{n+i}$$

for every $n \in \mathbb{N}$ and some polynomials $b_i(x) \in K[x]$, not all of them zero.

Proposition 11.2 (D-finiteness) *A sequence $(a_n) \subset K$, where the field K has characteristic* 0*, is* P*-recurrent if and only if its* OGF *$A(x) := \sum_{n \geq 1} a_n x^n$ is* D-finite*, which means that for some $k+1$, $k \in \mathbb{N}_0$, polynomials $p_0, \ldots, p_k \in K[x]$ the formal differential equation*

$$\sum_{i=0}^{k} p_i(x) A^{(i)}(x) = 0$$

holds.

If K has characteristic $p > 0$, one has that $A'(x) = 0$ whenever $A \in K[[x^p]]$.

Proposition 11.3 (algebr. $\Rightarrow$ D-fin.) *Let the field K have characteristic 0. Every algebraic formal power series A in $K[[x]]$ is* D*-finite.*

That A is algebraic means that $P(x, A(x)) = 0$ for some nonzero polynomial $P \in K[x, y]$. So the method that we demonstrated above on $C(x)$ works in general.

Chapter 12

*A definition of explicit (closed) enumerative formulae

Chapter 13

*Counting pattern-free permutations

Important combinatorial structures counted by C_n are permutations $\pi \in \mathcal{S}_{n-1}$ (i.e., with length $n-1$) avoiding any fixed permutation ρ with length 3. Since $|\mathcal{S}_3| = 6$, for $n = 4$ we indeed have for any ρ exactly $5 = C_4$ such permutations π because only one of the permutations in $\mathcal{S}_3$, namely ρ itself, contains ρ. But let us define precisely all these notions.

Chapter 14

*Counting $(1, 3, 2)$-free permutations

We consider the set $\mathcal{S}_{132}$ of 132-free permutations. We show that there is a size preserving bijection

$$F\colon \mathcal{S}_{132} \setminus \{\emptyset\} \to \mathcal{S}_{132} \times \mathcal{S}(1) \times \mathcal{S}_{132} \ .$$

Then we get for the OGF

$$A(x) := \sum_{\pi \in \mathcal{S}_{132}} x^{|\pi|} = \sum_{n=0}^{\infty} |\mathcal{S}_{132}(n)| x^n$$

the quadratic equation

$$A(x) - 1 = A(x) \cdot x \cdot A(x), \ \ \text{that is,} \ \ xA(x)^2 - A(x) + 1 = 0 \ .$$

It is better to write it, after multiplying by x, as

$$(xA(x))^2 - (xA(x)) + x = 0 \ .$$

Then clearly $xA(x) = C(x)$ and

$$A(x) = x^{-1}C(c) = \sum_{n=0}^{\infty} C_{n+1} x^n \ .$$

Thus

$$|\mathcal{S}_{132}(n)| = C_{n+1} = \frac{1}{n+1}\binom{2n}{n} \ .$$

We still have to describe the bijection F but this is not very hard. If $\pi = (a_1, \dots, a_n)$ with $n \in \mathbb{N}$ is a non-empty 132-free permutation, we take the unique $m \in [n]$ with $a_m = n$ and define

$$F(\pi) = (\pi_1,\, (1),\, \pi_2) := \big(\nu(a_1,\, \dots,\, a_{m-1}),\, (1),\, \nu(a_{m+1},\, \dots,\, a_n)\big)$$

(ν is the normalization operator of the previous chapter). It is clear that both π_1 and π_2 are 132-free and that $|\pi| = |\pi_1| + |(1)| + |\pi_2|$. So F is a size preserving map from $\mathcal{S}_{132}$ to $\mathcal{S}_{132} \times \mathcal{S}(1) \times \mathcal{S}_{132}$. Crucial is the existence of the reverse map G which we define for $\pi_1 = (a_1, \ldots, a_k)$ and $\pi_2 = (b_1, \ldots, b_l)$ in $\mathcal{S}_{132}$ by

$$G(\pi_1,\, (1),\, \pi_2) := \pi_1 \ominus (1) \ominus \pi_2$$

Chapter 15

*Counting $(1,2,3)$-free permutations

For $m, n \in \mathbb{N}$ let

$$a_n = |M_n| := |\{\pi \in \mathcal{S}_n \mid \pi \not\supset (1,\,2,\,3)\}|$$

be the number of $(1,2,3)$-free n-permutations, i.e., the number of the permutations of $1, 2, \ldots, n$ containing no increasing subsequence with length 3, and let

$$a_{n,\,m} = |M_{n,\,m}| := |\{\pi \in M_n \mid m(\pi) = m\}| \;.$$

For any $\pi \in \mathcal{S}_n$ differing from $(n, n-1, \ldots, 1)$ we define the parameter $m(\pi)$ by

$$m(\pi) := \min(\{j \in [n] \mid \exists\, i \in [j-1] \colon \pi(i) < \pi(j)\}) \;,$$

and by $m((n, n-1, \ldots, 1)) := n+1$ in the remaining case. Thus $m(\pi)$ records the end of the earliest $(1,2)$-copy in π, $a_n = \sum_{m=1}^{n+1} a_{n,m}$ and

$$M_n = \bigcup_{m=1}^{n+1} M_{n,\,m} \;.$$

For any $n \in \mathbb{N}$ we consider the bijection

$$F = F_n \colon X_n \to M_{n+1},\;\; X_n \subset M_n \times [n+1] \;,$$

that describes all insertions of the term $n+1$ in a $\pi \in M_n$ on a place $j \in [n+1]$ that produce a $\pi' \in M_{n+1}$. In more details, for any $\pi \in M_n$, $\pi = (a_1, \ldots, a_n)$, and any $j \in [n+1]$ we put

$$(\pi,\, j) \;\text{ in }\; X_n \iff (a_1,\, \ldots,\, a_{j-1},\, n+1,\, a_j,\, \ldots,\, a_n) =: \pi' = F(\pi,\, j) \in M_{n+1} \;.$$

Here $F(\pi, n+1) = (a_1, \ldots, a_n, n+1)$ and $F(\pi, 1) = (n+1, a_1, \ldots, a_n)$. It is easy to see that F is a bijection: the pair π, j is uniquely recovered from π', hence injectivity, and for every $\pi' \in M_{n+1}$ there is a $\pi \in M_n$ and a $j \in [n+1]$ such that $F(\pi, j) = \pi'$, simply delete $n+1$ from π' (which, crucially, never creates a copy of $(1,2,3)$), hence surjectivity.

The parameter $m(\pi)$ has two important properties: (i) it enables us to determine explicitly the sets X_n and (ii) $m(\pi') = m(F(\pi, j))$ can be explicitly computed from $m(\pi)$ and j; without property (ii), only with (i), the parameter $m(\pi)$ would not be really useful since then we could not capture explicitly by induction the working of $F = F_n$ for n running in $\mathbb{N}$. As for (i), it is easy to see that if $\pi \in M_n$ and $j \in [n+1]$ then

$$(\pi,\, j) \in X_n \iff j \le m(\pi)\ .$$

This justifies the definition that $m((n, n-1, \ldots, 1)) := n+1$ because $n+1$ can be inserted on any place in $(n, n-1, \ldots, 1)$ without creating a copy of $(1,2,3)$. As for (ii), it is easy to see that if $\pi \in M_n$ and $j \in [n+1]$ with $j \le m(\pi)$ then

$$m(\pi') = m(F(\pi,\, j)) = \left\{ \begin{array}{lll} j & \ldots & j \ge 2 \text{ and} \\ m(\pi)+1 & \ldots & j = 1\ . \end{array} \right.$$

It is not hard to express (i) and (ii) in the language of generating functions recording the numbers a_n and $a_{n,m}$. Let

$$A(x) := \sum_{n=1}^{\infty} a_n x^n \ \text{ and } \ A(x,\, y) := \sum_{n,\, m=1}^{\infty} a_n x^n y^m\ ,$$

so that $A(x) = A(x,1)$. Any permutation $\pi \in M_n$ with $m(\pi) = m \in [n+1]$, recorded by the monomial $x^n y^m$, produces by (i) and (ii) under the application of F_n exactly m permutations $\pi' \in M_{n+1}$, recorded by the monomials

$$x^{n+1}y^2 + x^{n+1}y^3 + \cdots + x^{n+1}y^m + x^{n+1}y^{m+1} = \frac{xy^2}{y-1}(x^n y^m - x^n \cdot 1)\ .$$

The permutations π' thus obtained, for π running in M_n and n running in $\mathbb{N}$, form the set

$$\bigcup_{n\ge 2} M_n = \bigcup_{n \ge 1} M_n \setminus \{(1)\}$$

and the permutation (1) is recorded by the monomial xy^2. Hence we obtain the identity

$$A(x,\, y) - xy^2 = \frac{xy^2}{y-1}\,(A(x,\, y) - A(x,\, 1))\ .$$

But how to solve this? We have only *one* equation for *two* unknown quantities $A(x,y)$ and $A(x,1)$. It seems we got stuck in a dead end. But there is a way out: we rewrite the equation to isolate $A(x,y)$ on one side as

$$A(x,\, y)\left(1 - \frac{xy^2}{y-1}\right) = xy^2 - A(x,\, 1)\frac{xy^2}{y-1}\ .$$

Or, better, as

$$A(x,\,y)\cdot(xy^2-y+1)=xy^2A(x,\,1)-xy^2(y-1)\;. \tag{15.1}$$

If we can substitute for y a power series $y=y(x)\in\mathbb{C}[[x]]$ such that $xy^2-y+1=0$, the left-hand side vanishes and we get one equation for one unknown quantity $A(x,1)$ which we then can solve for $A(x,1)$ (provided that xy^2 does not vanish). And $A(x,1)=A(x)$ is what we are interested in most. After finding $A(x,1)$ we can compute $A(x,y)$, but we are not so interested in this quantity.

We solve

$$xy^2-y+1=0$$

for y,

$$y=y(x)=(1/2x)\left(1\pm\sqrt{1-4x}\right)\;,$$

and substitute $y=y(x)=(1-\sqrt{1-4x})/2x=-\sum_{n\geq 0}\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1}x^n$ for y in Equation (15.1). We get that

$$0=xy^2A(x,\,1)-xy^2(y-1)$$

and, as $xy^2\neq 0$,

$$A(x)=A(x,\,1)=y-1=\sum_{n\geq 1}\frac{(-1)^n}{2}\binom{1/2}{n+1}4^{n+1}x^n\;.$$

Chapter 16

*Counting standard Young tableaux by the kernel method

In this chapter we closely follow pp. 582–585 of the MBM article [5]. A *partition* λ is any finite non-increasing sequence

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m)$$

with the *parts* $\lambda_i \in \mathbb{N}$. We write that $\lambda \vdash \sum_{i=1}^{m} \lambda_i =: n \in \mathbb{N}$ and say that λ *is a partition of* n. See the book [2] of G. Andrews for the beautiful enumerative theory of partitions. The *Ferrers diagram* $F(\lambda)$ *of* λ is the set of lattice points

$$F(\lambda) := \{(i,\, j) \mid 1 \leq i \leq m \wedge 1 \leq j \leq \lambda_i)\} \subset \mathbb{N}^2\ .$$

Let $\lambda \vdash n$. Any bijective map $T\colon F(\lambda) \to [n]$ such that

$$(i,\, j),\, (i,\, j') \in F(\lambda),\, j < j' \Rightarrow T(i,\, j) < T(i,\, j')$$

and

$$(i,\, j),\, (i',\, j) \in F(\lambda),\, i < i' \Rightarrow T(i,\, j) < T(i',\, j)$$

— T increases in every row and every column of the Ferrers diagram — is called the *standard Young tableau (with shape* λ*)*, briefly a *SYT*. For example, all sixteen SYTs with the shape $(3, 2, 1) \vdash 6$ are

$$\begin{array}{ccc}1&2&3\\4&5&\\6&&\end{array}\ ,\quad \begin{array}{ccc}1&2&3\\4&6&\\5&&\end{array}\ ,\quad \begin{array}{ccc}1&2&4\\3&5&\\6&&\end{array}\ ,\quad \begin{array}{ccc}1&2&4\\3&6&\\5&&\end{array}\ ,\quad \begin{array}{ccc}1&2&5\\3&4&\\6&&\end{array}\ ,\quad \begin{array}{ccc}1&2&5\\3&6&\\4&&\end{array}\ ,$$

$$\begin{array}{ccc}1&2&6\\3&4&\\5&&\end{array}\ ,\quad \begin{array}{ccc}1&2&6\\3&5&\\4&&\end{array}$$

and the eight SYTs obtained from them by flipping along the main diagonal. We derive by the kernel method the next interesting formula.

Theorem 16.1 *The number* f^λ *of* SYTs *with the shape* $\lambda \vdash n$ *is*

$$f^\lambda = \frac{n!}{\prod_{i=1}^m (\lambda_i - i + m)!} \prod_{1 \le i < j \le m} (\lambda_i - \lambda_j - i + j) \ .$$

In our example

$$\frac{6!}{(3-1+3)! \cdot (2-2+3)! \cdot (1-3+3)!} \cdot (3-2-1+2)(3-1-1+3)(2-1-2+3)$$

indeed equals $2 \cdot 4 \cdot 2 \cdot 6!/5! \cdot 3! \cdot 1! = 2 \cdot 4 \cdot 2 = 16$.

Proof. Let $m \ge 2$ (for $m = 1$ the formula holds trivially). For $j = 2, \ldots, m$ let

$$F(u) := \sum_{\lambda_1 \ge \cdots \ge \lambda_m \ge 0} f^\lambda \prod_{i=1}^m u_i^{\lambda_i} \ \text{ and } \ F_j(u) := \mathrm{D}_{j-1,\,j}(F(u))$$

be, respectively, the OGF of the numbers f^λ for partitions λ with at most m parts (we fill in the dummy zero parts) and the OGF of the numbers f^λ for the partitions with at most m parts and equal parts $\lambda_{j-1} = \lambda_j$. Here u is the m-tuple of variables $u_1, \ldots, u_m$ and $\mathrm{D}_{j-1,\,j}$ denotes the operation of the $(j-1, j)$-*diagonal*, which keeps from the argument power series only the terms such that u_{j-1} and u_j have equal exponents.

The functional equation

$$F(u) = 1 + u_1 F(u) + \sum_{j=2}^m u_j (F(u) - F_j(u))$$

records all possible ways to extend a fixed SYT T with the shape $\lambda \vdash n$ to a SYT T' with the shape $\lambda' \vdash n+1$ such that $F(\lambda') \supset F(\lambda)$ and $F(\lambda') \setminus F(\lambda) = \{(k,l)\}$, $T' \mid F(\lambda) = T$ and $T'(k,l) = n+1$. On the right-hand side, the term 1 corresponds to the empty SYT (the only one which does not arise by these extensions), the term $u_1 F(u)$ corresponds to adding $(k,l) = (1, \lambda_1 + 1)$ to $F(\lambda)$, which is always possible, and the summand $u_j(F(u) - F_j(u))$ corresponds to adding $(k,l) = (j, \lambda_j + 1)$ to $F(\lambda)$, which is possible iff $\lambda_{j-1} > \lambda_j$. We isolate the kernel $K(u)$ and write the equation as

$$\underbrace{\left(1 - \sum_{j=1}^m u_j\right)}_{K(u)} F(u) = 1 - \sum_{j=2}^m u_j F_j(u) \ .$$

The ultimate form of the equation is obtained by multiplying it by $M(u) := u_1^{m-1} u_2^{m-2} \ldots u_{m-1}^1 u_m^0$:

$$K(u) M(u) F(u) = M(u) - \sum_{j=2}^m (u_j M(u)) \cdot F_j(u) \ .$$

The point of the last transformation is that in the monomial $u_jM(u)$ the variables u_{j-1} and u_j have equal exponents, and thus so they have (by the definition of $F_j(u)$) also in each term of the summand power series $(u_jM(u)) \cdot F_j(u)$. Thus, if we denote by $\varepsilon(\sigma)$ the sign ± 1 of a permutation $\sigma \in \mathcal{S}_m$ and by $\sigma(P)$ the action of σ on $P \in \mathbb{C}[[u_1, \ldots, u_m]]$ by permuting the m variables, we get by applying $\sigma(\cdot)$ on the last displayed equation, multiplying it by $\varepsilon(\sigma)$ and summing the results over all $\sigma \in \mathcal{S}_m$ the equation

$$\sum_{\sigma \in \mathcal{S}_m} \varepsilon(\sigma)\sigma(M(u)F(u)) = \frac{1}{K(u)} \sum_{\sigma \in \mathcal{S}_m} \varepsilon(\sigma)\sigma(M(u)) \ . \tag{SYT}$$

It follows from the facts that $\sigma(K(u)) = K(u)$ (and that $\sigma(\cdot)$ is a ring automorphism of $\mathbb{C}[[u_1, \ldots, u_m]]$) and that the term $\sum_{j=2}^{m}(u_jM(u)) \cdot F_j(u)$ cancels out in the signed sum. To explain the last fact, we take any $j \in \{2, \ldots, m\}$ and denote by $\sigma_j \in \mathcal{S}_m$ the transposition that exchanges $j-1$ and j. Since $m \geq 2$, it is possible to partition $\mathcal{S}_m$ in pairs $\{\sigma, \sigma'\}$ such that $\sigma = \sigma_j\sigma'$ (and $\sigma' = \sigma_j\sigma$). Since here $\varepsilon(\sigma) \neq \varepsilon(\sigma')$ and $(u_jM(u)) \cdot F_j(u)$ is fixed by σ_j, we see that

$$\sum_{\sigma \in \mathcal{S}_m} \varepsilon(\sigma)\sigma\big((u_jM(u)) \cdot F_j(u)\big) = 0 \ .$$

We obtain the numbers f^λ by comparing coefficients on both sides of (SYT):

$$f^\lambda = [u_1^{m-1+\lambda_1} \ldots u_m^{0+\lambda_m}] \frac{1}{K(u)} \sum_{\sigma \in \mathcal{S}_m} \varepsilon(\sigma)\sigma(M(u)) \ .$$

$\square$

Chapter 17

*Counting $(1, 2, \ldots, m+1)$-free permutations — an explicit (closed) formula

Chapter 18

*Counting $(1, 2, \ldots, m+1)$-free permutations — a determinantal formula for the Bessel generating function

Chapter 19

*Counting labeled k-regular graphs — an explicit (closed) formula

Chapter 20

*Counting labeled k-regular graphs — a way to P-recursiveness via symmetric functions

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Index

Exam questions.

1. Proofs no. 1 and 2 that (C_n) is not a LRS.
2. Proofs no. 3 and 4 that (C_n) is not a LRS.
3. Derive that $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ by OGF.
4. Prove that $C_n = \frac{1}{n}\binom{2n-2}{n-1}$ combinatorially.
5. Prove that $C_n \leq 8^{n-1}/n^2$ for every $n \in \mathbb{N}$.
6. Prove that C_{n+1} is the number of $(1, 3, 2)$-free permutations in $\mathcal{S}_n$.
7. Prove that C_{n+1} is the number of $(1, 2, 3)$-free permutations in $\mathcal{S}_n$.