Davenport–Schinzel Sequences and Their Applications in Combinatorial Geometry and Graph Theory

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Overview

- DS sequences and extremal functions $\lambda_s(n)$
- History — linear and nonlinear bounds
- Applications in combinatorial geometry
- Applications in graph theory
- Generalization of $\lambda_s(n)$ to $\mathrm{Ex}(u, n)$
- Generalization to coloured trees

M. Klazar, Generalized Davenport–Schinzel sequences: results, problems, and applications, *Integers* **2** (2002), A11, 39 pages.
Available on `http://www.integers-ejcnt.org/`

Definition of DS sequences

A sequence $u = a_1 a_2 \ldots a_l$ of some symbols is an s-*DS sequence*, where $s \in \mathrm{IN} = \{1, 2, \ldots\}$ is a parameter, if two conditions are satisfied.

- For every i we have $a_i \neq a_{i+1}$. That is, u has no immediate repetitions.
- There are no s indices $1 \leq i_1 < i_2 < \ldots < i_s \leq l$ and no two distinct symbols a and b such that $a = a_{i_1} = a_{i_3} = a_{i_5} = \cdots$ and $b = a_{i_2} = a_{i_4} = a_{i_6} = \cdots$. In other words, u has no alternating subsequence of length s.

Examples. $u = a\ c\ b\ d\ c\ a\ d$ is a 5-DS sequence but is not a 4-DS sequence. $u = 2\ 2\ 2\ 1\ 4\ 7\ 1\ 5\ 6\ 6\ 5\ 1$ has immediate repetitions and thus is not an s-DS sequence for any s. “Contracting” the immediate repetitions in u we get a 4-DS sequence $u' = 2\ 1\ 4\ 7\ 1\ 5\ 6\ 5\ 1$ which is not a 3-DS sequence.

Extremal functions $\lambda_s(n)$

For $n \in \mathbb{N}$ and $s \geq -1$ we define the extremal function $\lambda_s(n)$ as

$\lambda_s(n) =$ the maximum length of an $s+2$-DS sequence u with $\leq n$ symbols.

Why the $+2$ shift in s? Standard notation due to an important geometric application.

Trivial examples. $\lambda_{-1}(n) = 0$ (only $u = \emptyset$), $\lambda_0(n) = 1$ (only $u = 1$), and $\lambda_1(n) = n$ (the longest sequence is $u = 1, 2, \ldots, n$).

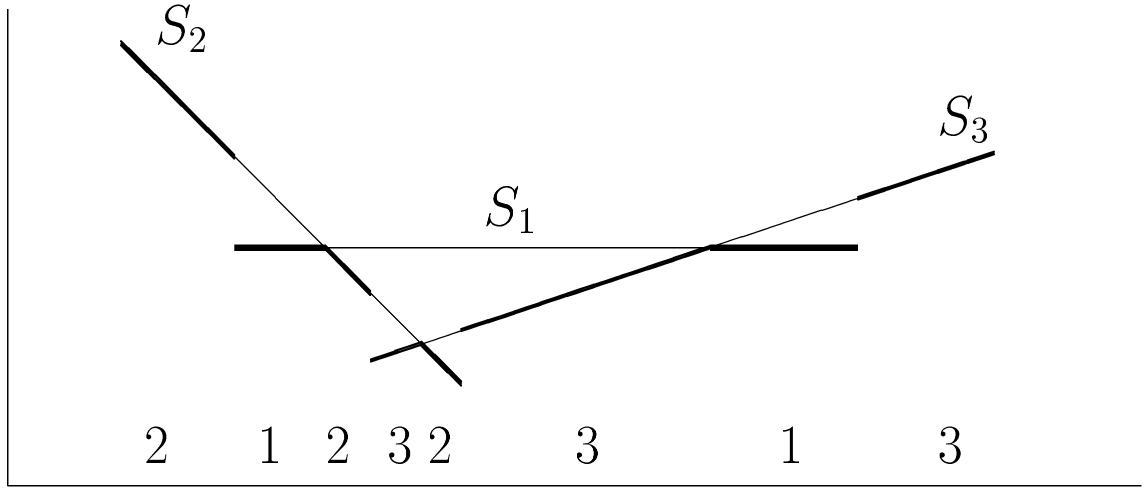
Proposition. $\lambda_2(n) = 2n - 1$ for all $n \in \mathbb{N}$.
Proof. $\lambda_2(n) \geq 2n - 1$ because of, for example,

$$u = 1, 2, \ldots, n-1, n, n-1, \ldots, 2, 1.$$

We prove $\lambda_2(n) \leq 2n - 1$ by induction on n. Let u, $|u| = \lambda_2(n)$, be a 4-DS sequence with n symbols. Some a appears in u only once: consider the a's inbetween two closest occurrences of a symbol b. We delete a and possibly its neighbour from u and apply induction. $\square$

$\lambda_3(n)$ and plane segments

Consider a system S of n straight segments in the plane, no two overlapping and none vertical, and their lower envelope:



We call $u = 2, 1, 2, 3, 2, 3, 1, 3$ the *minimazing sequence* of S. A subsequence of the type $\ldots a \ldots b \ldots a \ldots b \ldots a \ldots$ in $u \Rightarrow$ two segments intersects in two points. Hence u is a 5-DS sequence and $|u| \leq \lambda_3(n)$.

What is $\lambda_3(n)$?

Proposition. For $n \to \infty$,

$$\lambda_3(n) \ll n \log n$$

where $\ll$ is synonymous to the O-notation.

Proof. Let u, $|u| = \lambda_3(n)$, be the longest 5-DS sequence using symbols $1, 2, \ldots, n$ and k_i be the number of occurrences of $i = 1, 2, \ldots, n$ in u. Key observation:

$$|u| - k_i - 2 = \lambda_3(n) - k_i - 2 \leq \lambda_3(n-1),$$

because we may have $\ldots aia \ldots$ in u only if this is the first or the last occurrence of i. Since $k_1 + k_2 + \cdots + k_n = |u|$, summing over all i we get $(n-1)\lambda_3(n) - 2n \leq n\lambda_3(n-1)$ which is the same as

$$\frac{\lambda_3(n)}{n} - \frac{\lambda_3(n-1)}{n-1} \leq \frac{2}{n-1}.$$

Summing these inequalities for $2, 3, \ldots, n$ we get $\lambda_3(n) \leq n(1 + 2(1^{-1} + 2^{-1} + \cdots + (n-1)^{-1})) \ll n \log n$. $\square$

History

- 1965 — **Davenport & Schinzel** introduced DS-sequences and functions $\lambda_s(n)$ (in a different notation) and proved that $\lambda_2(n) = 2n-1$, $\lambda_3(n) \ll n \log n$, and $\lambda_s(n) \ll n \exp(c(\log n)^{1/2})$ where $c > 0$ depends only on s.

- 1971 — **Davenport(†)** proved that $\lambda_3(n) \ll n \log n / \log\log n$ and that $\lim \lambda_3(n)/n \geq 8$.

- 1974 — **Szemerédi** proved that $\lambda_s(n) \ll_s n \log^* n$, where $\log^* n$ is the smallest number of 'logs' for which $\log\log \ldots \log n \leq 1$.

- 1970–76 — Particular bounds and enumerative results (these only for 4-DS sequences) were found by **Burkowski, Dirksen, Dobson, Ecklund, MacDonald, Mills, Mullin, Peterkin, Rennie, Roselle, and Stanton.**

However, the question if $\lambda_s(n) \ll n$ for $s > 2$ remained open.

History

- 1985 — **Atallah** rediscovered DS sequences in the context of computational geometry.
- 1986 — **Hart & Sharir** proved, using compression schemes, that $n\alpha(n) \ll \lambda_3(n) \ll n\alpha(n)$ where $\alpha(n)$ is the *inverse Ackermann function* (we review its definition later). Thus $\lambda_s(n) \not\ll n$ for $s > 2$.
- 1988 — **Komjáth** proved $\lambda_3(n) \gg n\alpha(n)$ working only with sequences.
- 1988 — **Wiernik & Sharir** constructed systems of n plane segments whose lower envelopes have $\gg n\alpha(n)$ portions.
- 1987–88 — **Sharir** proved lower and upper bounds on $\lambda_s(n)$ in terms of $\alpha(n)$ for general s.
- 1989 — **Agarwal, Sharir & Shor** proved that $n2^{\alpha(n)} \ll \lambda_4(n) \ll n2^{\alpha(n)}$ and ($s > 4$)

$$\lambda_s(n) \approx n \cdot 2^{\alpha(n)^{(s-2)/2}}.$$

History, and the definition of $\alpha(n)$

- 1995 — the book **Sharir & Agarwal**, Davenport-Schinzel Sequences and Their Geometric Applications, Cambridge University Press, Cambridge UK.
- 1986–present — many applications of DS sequences in computational geometry.

The definition of $\alpha(n)$
One defines first the *Ackermann function* $A(n)$ as $A(n) = F_n(n)$ where $F_1(n) = 2n$, $F_2(n) = 2^n$ and, in general,

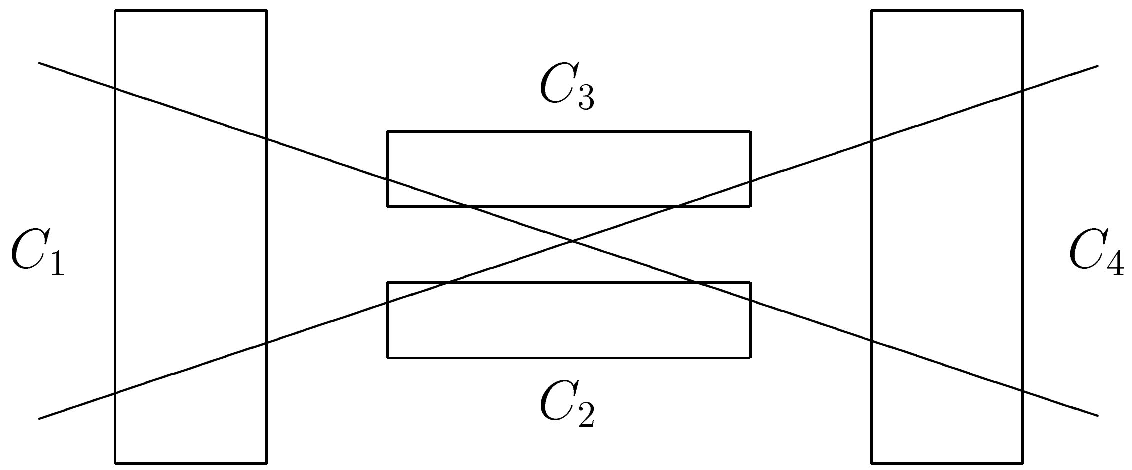
$$F_{i+1}(n) = F_i(F_i(\ldots F_i(1)\ldots)) \text{ with } n \text{ iterations of } F_i.$$

Then $\alpha(n) = \min\{m \in \mathrm{IN} : \ A(m) \geq n\}$.

The initial values
$\alpha(1) = \alpha(2) = 1$, $\alpha(3) = \alpha(4) = 2$, $\alpha(5) = \alpha(6) = \cdots = \alpha(16) = 3$, and $\alpha(n) = 4$ for $17 \leq n \leq 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}$ with $2^{16} = 65536$ 2's.

Two geometric applications of $\lambda_2(n)$

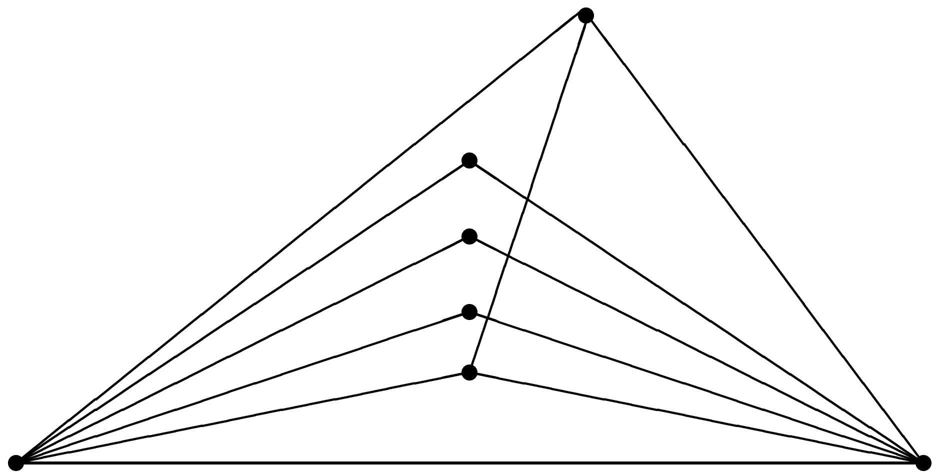
stabbing of convex sets
Let $C_1, C_2, \ldots, C_n$ be compact and convex plane sets, mutually disjoint. In how many distinct ways can one intersect all C_i by a line? In this picture we get two transversal permutations, $1, 2, 3, 4$ (or $4, 3, 2, 1$) and $1, 3, 2, 4$ (or $4, 2, 3, 1$):



1990 — **Edelsbrunner & Sharir** proved that the maximum number of transversal permutations is $2n - 2$ ($n \geq 4$). (**Katchalski, Lewis & Zaks**, 1985: $\leq \binom{n}{2}$, **Wenger**, 1986: $\leq 6n + 6$.)

Two geometric applications of $\lambda_2(n)$

geometric graphs with no two "parallel" edges
Two plane segments are "parallel" if they are disjoint and form the opposite sides of a convex quadrilateral. This geometric graph has $n = 7$ vertices, $2n - 2$ edges, and it has no two "parallel" edges:



1998 — **Valtr** proved that $2n - 2$ is maximum possible (conjectured by Kupitz in 1979, in 1998 **Katchalski & Last** proved the $\leq 2n - 1$ bound).

Three problems on $\lambda_s(n)$

Problem 1. Is it true that *every* 5-DS sequence can be realized as a minimazing sequence of an appropriate system of plane segments?

(Every 5-DS sequence can be easily realized by pseudosegments.)

Problem 2. It follows from a lower bound of **Roselle & Stanton** and an upper bound due to **Davenport & Schinzel** that

$$\frac{n^3(1+o(1))}{3} < \lambda_n(n) < \frac{n^3(1+o(1))}{2}.$$

What is the precise asymptotics of $\lambda_n(n)$?

Problem 3. Does the limit

$$\lim_{n\to\infty} \frac{\lambda_3(n)}{n\alpha(n)}$$

exist?

If it exists, it lies in the interval $[1, 2]$ (**Wiernik & Sharir** and **Klazar**).

Extremal functions $\mathrm{Ex}(u, n)$

Functions $\lambda_s(n)$ are defined by forbidding alternating subsequences of two symbols. What about more symbols? Let u and v be two finite sequences of some symbols.

- u and v are *isomorphic* — have the same length and differ only by an injective renaming of their symbols. For example, $a\ b\ b\ a\ c\ d\ a$ and $b\ 2\ 2\ b\ 4\ 3\ b$ are isomorphic.
- u is *contained* in v, in symbols $u \prec v$ — sequence u is isomorphic to a subsequence of v.
- v is *u-free* — $v \not\succ u$.
- $u = a_1 a_2 \ldots a_l$ is *k-sparse* — $a_i = a_j, i > j$, implies $i - j \geq k$. So 2-sparseness = no immediate repetition.

We generalize $\lambda_s(n)$ to $\mathrm{Ex}(u, n)$. Suppose the sequence u uses exactly k symbols. Let $|v|$ be the length of a sequence v.

$$\mathrm{Ex}(u, n) = \max\{|v| : \ v \text{ is } k\text{-sparse, } u\text{-free, and uses } \leq n \text{ symbols}\}.$$

Examples of $\mathrm{Ex}(u, n)$

Examples. $\lambda_2(n) = \mathrm{Ex}(abab, n)$ and $\lambda_3(n) = \mathrm{Ex}(ababa, n)$. Let $a^i = aa \ldots a$ with i a's. Then $\mathrm{Ex}(a^i, n) = (i-1)n$. $\mathrm{Ex}(adabca, n) = 2n$ ($n \geq 4$).

Proposition. $\mathrm{Ex}(abba, n) = 3n - 2$ for all $n \in \mathbb{N}$.

Proof. We have $\mathrm{Ex}(abba, n) \geq 3n - 2$ because of the sequences like

$$a\ b\ a\ b\ c\ b\ c\ d\ c\ d\ e\ d\ e.$$

Now suppose v is 2-sparse, $abba$-free, and uses $\leq n$ symbols. If some symbol a appears in v at least $4\times$ $\Rightarrow$ another symbol b appears in v only once. This is because $v \not\succ abba$ implies that in

$$\ldots a \ldots a \ldots b \ldots a \ldots a \ldots.$$

there cannot be other b. Deleting b and possibly its neighbour we get $|v| \leq 3(n-1) - 2 + 2 = 3n - 3$. Thus every symbol appears in v at most $3\times$ and $|v| \leq 3n$, which is improved by the end elements to $|v| \leq 3n - 2$. $\square$

Examples of $\mathrm{Ex}(u,n)$

Proposition. Let u be any sequence using exactly k symbols. Then $\mathrm{Ex}(u,n) < k\big((|u|-1)\binom{n}{k}+1\big) \ll n^k$.

Proof. Suppose v is k-sparse, uses $\leq n$ symbols, and $|v| \geq$ the RHS. Split v in $(|u|-1)\binom{n}{k}+1$ intervals of length k and apply the pigeonhole principle. $\square$

Proposition. We have $\mathrm{Ex}(abcabc,n) \ll n$. In general,

$$\mathrm{Ex}(a_1a_2\ldots a_ka_1a_2\ldots a_k,n) \ll n.$$

Proof. Let $L = \mathrm{Ex}(abcabc,16)+1$. Suppose v is 3-sparse, uses $\leq n$ symbols, and $|v| \geq (2n+1)L$. We prove that $v \succ abcabc$. We write $v = I_1I_2\ldots I_{2n+1}J$ with $|I_i| = L$. Some I_r contains neither the first nor the last appearance of any symbol. If I_r contains at most 16 distinct symbols, $I_r \succ abcabc$. So I_r contains at least 17 distinct symbols. These appear also before and after I_r. Applying twice the Erdős–Szekeres lemma, we deduce that v has a subsequence $A_1A_2A_3$ such that each A_i is xyz or zyx. Thus $v \succ abcabc$. $\square$

Short history

- 1992 — **Adamec, Klazar & Valtr** introduced $\mathrm{Ex}(u,n)$ and proved that $\mathrm{Ex}(a^ib^ia^ib^i,n) \ll n$ for every $i \in \mathbb{N}$. Thus, if u uses ≤ 2 symbols, $\mathrm{Ex}(u,n) \ll n \Leftrightarrow ababa \not\prec u$.

- 1992 — **Klazar** proved the general almost linear bound

$$\mathrm{Ex}(u,n) < c_1 \cdot n \cdot c_2^{\alpha(n)^{c_3}}$$

where the constants $c_i > 1$ depend only on u.

- 1994 — **Klazar & Valtr** proved that for many u one has the linear bound $\mathrm{Ex}(u,n) \ll n$. In particular, for the N-shaped sequence

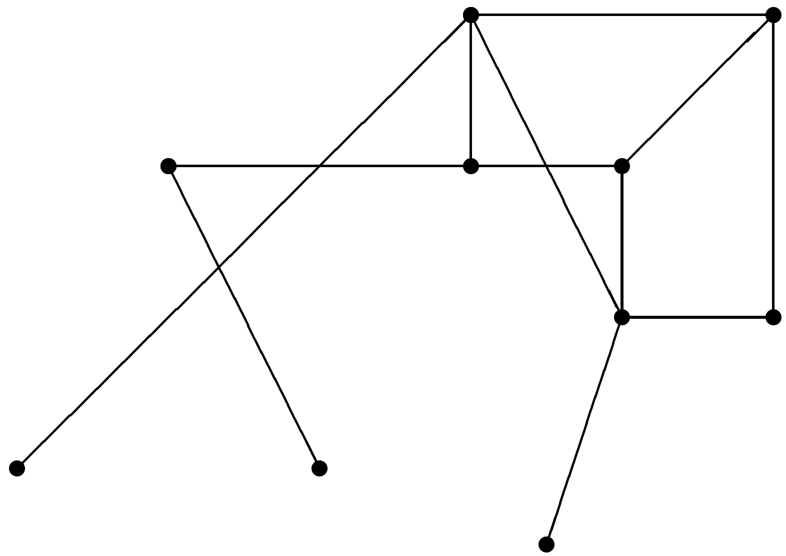
$$u_N(k,i) = 1^i 2^i \ldots (k-1)^i k^i (k-1)^i \ldots 2^i 1^i 2^i \ldots (k-1)^i k^i$$

(recall that j^i abbreviates the sequence $jj\ldots j$ of length i) one has $\mathrm{Ex}(u_N(k,i),n) \ll n$ for every k and i.

- 1997–present — **Klazar, Valtr** introduced some generalizations to trees.

A geometric application of $\mathrm{Ex}(u_N(k,i),n)$

In a geometric graph G, vertices are points and edges are straight plane segments:



1998 — **Valtr** proved that every G with n vertices and no k pairwise "parallel" edges has $\ll_k n$ edges. From this he derived that every G with n vertices and no k pairwise crossing edges has $\ll_k n \log n$ edges. (For $k \leq 3$ one has even $\ll n$ edges, the case $k = 2$ gives planar graphs and the case $k = 3$ was proved by **Agarwal et al.** in 1997.)

How to prove $\mathrm{Ex}(u_N(k,i),n) \ll n$

We mention two main results proved in 1994 by **Klazar & Valtr**.

Theorem A. If the sequences $u = u_1aau_2$ and v do not share symbols, then

$$\mathrm{Ex}(u_1avau_2, n) \ll \mathrm{Ex}(v, \mathrm{Ex}(u,n)).$$

Theorem B. If $u = u_1aau_2a$, $i \in \mathbb{N}$, and the symbol b does not appear in u, then

$$\mathrm{Ex}(u_1ab^iau_2ab^i, n) \ll \mathrm{Ex}(u,n).$$

Applications. 1. If $u \not\succ abab$ then $\mathrm{Ex}(u,n) \ll n$. 2. For every $i, k \in \mathbb{N}$ one has $\mathrm{Ex}(u_N(k,i),n) \ll n$.

Proof. 1. Then $u = a^{i_1}u_1a^{i_2}u_2a^{i_3}\dots a^{i_{r-1}}u_ra^{i_r}$ where $i_j \in \mathbb{N}_0$, the u_j's have no common symbol and no u_j uses a. Apply A and induction on the u_j's. 2. Start from a^i and apply B: aa, $abbab$, $abccbabc$, $abcddcbabcd$, $\dots$ $\square$

Three problems on $\mathrm{Ex}(u, n)$

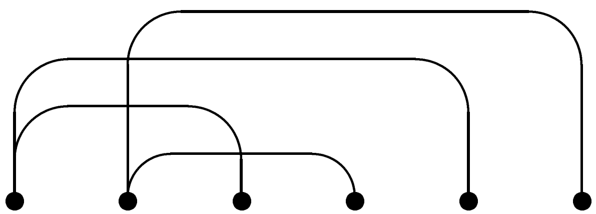
Problem 4. Is it true that $\mathrm{Ex}(acababcb, n) \ll n$?
(How to remember this sequence: $abab \to aababb \to acababcb$. It is one of the three maximal 2-sparse $ababa$-free sequences over $\{a, b, c\}$. The other two sequences, $ababcbc$ and $abcbabc = u_N(3, 1)$, have $\ll n$ extremal functions. This follows by A and B.)

Problem 5. Is it true that $\mathrm{Ex}(ab^2a^2b^2a, n) \ll n\alpha(n)$?
(This would imply, by a result of **Adamec, Klazar & Valtr**, that $\mathrm{Ex}(a^ib^ia^ib^ia^i, n) \ll n\alpha(n)$ for every $i \in \mathbb{N}$. It is trivial that $\mathrm{Ex}(a^ib^ia^ib^ia^i, n) \geq \lambda_3(n) \gg n\alpha(n)$.)

Problem 6. Is it true that for every $c \in \mathbb{N}$ there is an $ababa$-free sequence u such that $\mathrm{Ex}(u, n) \gg n \cdot 2^{\alpha(n)^c}$?
(The 1992 bound of **Klazar** says that $\mathrm{Ex}(u, n) \ll n \cdot 2^{\alpha(n)^c}$ for every u with a constant $c = c(u)$. Thus we ask if the growth rates of $\mathrm{Ex}(u, n)$ for $ababa$-free u are cofinal among all extremal functions. It is true for $c = 1$ with $u = abcbadadbecfcfedef$.)

An extremal problem for ordered graphs

Let $G = (\{1, 2, \ldots, n\}, E)$ be a graph such that for no six vertices $1 \le v_1 < v_2 < \ldots < v_6 \le n$ the pairs $\{v_1, v_3\}$, $\{v_1, v_5\}$,$\{v_2, v_4\}$, and $\{v_2, v_6\}$ are edges of G. In other words, G has no *ordered* subgraph of the form



What is the maximum possible number $g(n)$ of edges in G ?

2001 — **Klazar** realized that

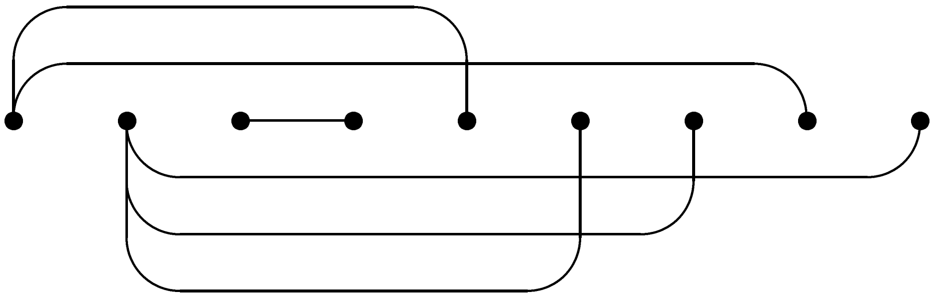
$$n\alpha(n) \ll \lambda_3(n) \ll g(n) \ll \lambda_6(n) \ll n \cdot 2^{(1+o(1))\alpha(n)^2}.$$

1992 — **Füredi & Hajnal** investigated similar problems for ordered bipartite graphs and applied DS sequences as well.

Develop the extremal theory of ordered graphs! It is nonexistent.

More generally: forbidden star forests

The previous upper bound on $g(n)$ generalizes to the situation when G does not contain, as an ordered subgraph, a fixed *star forest*. This is a graph $F = (\{1, 2, \ldots, m\}, E)$ such that $\{1, 2, \ldots, m\} = A \cup B$, $A < B$, A and B are independent sets in F, and $\deg(v) = 1$ for every $v \in B$. So F may look like the previous graph or like



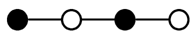
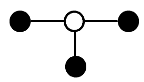
Theorem. (**Klazar**, 2001). Let F be a star forest. If $G = (\{1, 2, \ldots, n\}, E)$ does not contain F as an ordered subgraph, then

$$|E| < n \cdot c_1^{\alpha(n)^{c_2}}$$

where $c_i = c_i(F) > 1$ are constants.

A tree generalization of $\lambda_2(n)$

Sequences can be regarded as coloured paths. Coloured paths are a special case of coloured trees. We define $T(abab, n)$ = the maximum $|V|$ such that there is a *tree* $T = (V, E)$ and a vertex n-colouring $f : V \to \{1, 2, \dots, n\}$ satisfying three conditions.

1. f is proper (= no monochromatic edge).
2. (T, f) contains no subdivision of the properly 2-coloured 4-vertex path  .
3. (T, f) contains no subdivision of the properly 2-coloured 4-vertex star  .

$T(abab, n)$ generalizes $\lambda_2(n)$: 1 extends the no-immediate-repetition condition and 2 extends $u \not\succ abab$. Condition 3, which is void for sequences, takes care of large stars (without 3 we would have $T(abab, n) = \infty$). So $T(abab, n) \geq \lambda_2(n)$.

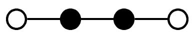
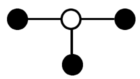
What is $T(abab, n)$?

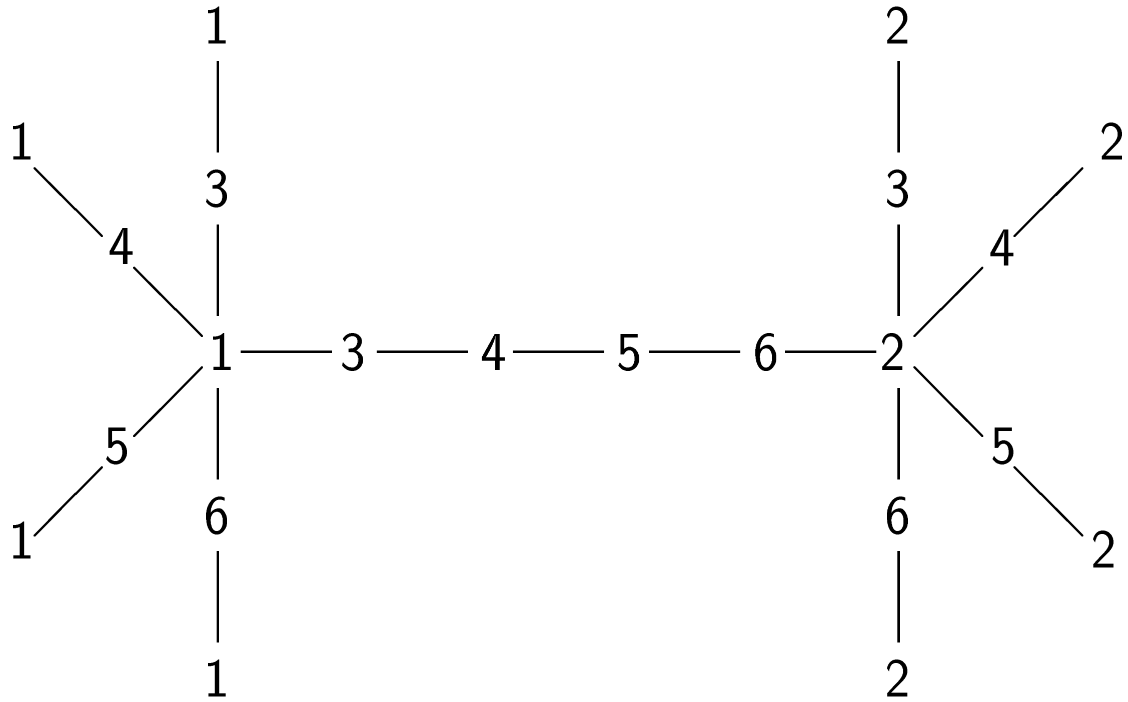
$T(abab, n) = \lambda_2(n) = 2n - 1$

Theorem. (**Klazar**, 1999). $T(abab, n) = 2n - 1$ for all $n \in \mathbb{N}$.

Proof. Note that $T(abab, n) \geq 2n - 1$ not only because of sequences but that every tree with $2n - 1$ vertices can be vertex-coloured by n colours so that 1, 2, and 3 hold. Let (T, f), $f : V \to \{1, 2, \ldots, n\}$, satisfy 1, 2, and 3. We prove that $|V(T)| \leq 2n - 1$ by induction on n and the number of vertices with degree ≥ 3. There exists a vertex w, $\deg(w) \geq 3$, such that $T - w$ has $r \geq 2$ path components $P_1, \ldots, P_r$ and one possibly non-path component. We distinguish two cases.
1. $f|(P_1 \cup \ldots \cup P_r \cup \{w\})$ has a repeated colour. Then a vertex $u \in P_i$ has a unique colour (in the whole (T, f)) or two vertices $u_1 \in P_i$ and $u_2 \in P_j$, $i \neq j$, share a unique colour. Delete u (and possibly a neighbour of u) or delete u_1 and u_2, and use induction on n.
2. $f|(P_1 \cup \ldots \cup P_r \cup \{w\})$ has no repeated colour. It follows that the vertices in $P_1 \cup \ldots \cup P_r$ can be assembled in one path P (that is joined to w) so that no subdivision of $\bullet\!-\!\circ\!-\!\bullet\!-\!\circ$ is created. Use induction on the number of vertices with degree ≥ 3. (Now $\deg(w) = 2$.) $\square$

What about $T(abba, n)$?

Recall that $\mathrm{Ex}(abba, n) = 3n - 2$ (slide 14). $T(abba, n)$ is the maximum number of vertices in a tree that can be properly n-coloured so that all subdivisions of  and  are avoided. This example with $n = 6$ shows that $T(abba, n) \geq 5n - 8$:



$T(abba, n) \ll n$

Theorem. (**Valtr**, 1999). $T(a^i b^i a^i, n) \leq 24in$ for all $i, n \in \mathbb{N}$.

Thus $5n - 8 \leq T(abba, n) \leq 48n$.

Problem 7. What is $T(abba, n)$? Determine this function exactly.

Recall that $\mathrm{Ex}(ab^2a^2b, n) \ll n$ (slide 16). In fact, **Klazar** proved in 1996 that $\mathrm{Ex}(ab^2a^2b, n) \leq 8n - 7$. **Valtr** proved in 1999 that

$$T(ab^2a^2b, n) \gg n\alpha(n).$$

Despite the coincidence on $u = abab$, $\mathrm{Ex}(u, n)$ and $T(u, n)$ behave rather differently. **Valtr** obtained many results on $T(u, n)$ and other tree generalizations of $\mathrm{Ex}(u, n)$ (to appear in *Combinatorica*). There are applications of DS sequences to enumerative problems and a generalization of the sequential containment to hypergraphs but we stop here.

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