# Combinatorial Counting 2026

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# Introduction

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# Chapter 1

# The symbolic method, the Catalan numbers and Pólya's theorem

## 1.1 The symbolic method. Proper bracketings. Catalan numbers

### The symbolic method

The fundamental *symbolic method in enumerative combinatorics*, briefly the *symbolic method*, addresses the problem to enumerate — both by exact counting formulas and by approximate asymptotic formulas — a given countable (infinite) set $A$ of combinatorial objects. $A$ is equipped with a *size function* $s\colon A \to \mathbb{N}_0$ $(= \{0, 1, \dots\})$ and it is assumed that for every $n \in \mathbb{N}_0$ the set of size $n$ objects in $A$,

$$A_n = \{a \in A\colon\ s(a) = n\}\,,$$

is finite. We assign to $A$ and $s$ the generating function, briefly gf,

$$A(x) = \textstyle\sum_{a \in A} x^{s(a)} = \sum_{n \ge 0} |A_n| x^n \quad (\in \mathbb{C}[[x]])\,,$$

where $|X|$ denotes the number of elements of a finite set $X$. $A(x)$ is a *formal power series*, briefly fps. Set-theoretically, it is a map

$$A(x)\colon \mathbb{N}_0 \to \mathbb{C},\ A(x)(n) = |A_n|\,.$$

We review basic algebraic properties of fps in the next section.

Besides the function $s(a)$ we often have a *weight function* $h\colon A \to R$, where $R$ is a ring. The weighted version of $A(x)$ is then

$$A_h(x) = \textstyle\sum_{n \ge 0} \big(\sum_{a \in A_n} h(a)\big) x^n \quad (\in R[[x]])\,.$$

The symbolic method works in two phases. In the *first formal phase* relations between objects in $A$ are translated into algebraic, differential or functional equations for the gf $A(x)$ or $A_h(x)$ and auxiliary gf. Gf with several variables are frequently used. By formal (algebraic) manipulations with involved fps we obtain in this phase counting formulas for the coefficients $|A_n|$ or $\sum_{a \in A_n} h(a)$. In the *second analytic phase*, we obtain by means of analytic methods from the equations derived in the first phase asymptotic formulas for the mentioned coefficients.

### Proper bracketings

We exemplify the symbolic method by enumerating the set $A$ of proper bracketings. A *proper bracketing* is a word $b = b_1 b_2 \dots b_{2n}$ over the two-element alphabet $\{(,\ )\}$, consisting of the left bracket ( and the right bracket ), with length $2n$, $n \in \mathbb{N}_0$, and with the following structure. For $n = 0$ we set $b = \emptyset$, and for $n \in \mathbb{N}$, 1 and 2 below hold.

1. $b$ has $n$ (s and $n$ )s.
2. There is a partition $P_b$ of the index set $[2n] = \{1, 2, \dots, 2n\}$ in $n$ two-element blocks such that for every block $B = \{i < j\} \in P_b$ we have $b_i =$ ( and $b_j =$), and for every two distinct blocks $B, C \in P_b$ we have neither $\min B < \min C < \max B < \max C$ nor $\min C < \min B < \max C < \max B$; we say that $B$ and $C$ *do not cross*.

Thus $P_b$ belongs to the family of so called non-crossing partitions. The following proposition is a fundamental mathematical result; the syntax of propositional logic builds on it.

**Proposition 1.1** *The partition $P_b$ is uniquely determined by the word $b$.*

*Proof.* Let $b = b_1 \dots b_{2n} \in A$ with length $2n \geq 2$; for $b = \emptyset$ the result trivially holds. We consider the function $f \colon [2n] \to \mathbb{N}_0$,

$$f(j) = \text{the number of } i \in [j] \text{ with } b_i = ( \; - \text{ the number of } i \in [j] \text{ with } b_i = )\,.$$

We show by induction on $n$ that $P_b$ is unique. Let $B = \{1, i_0\} \in P_b$ be the first block of the partition $P_b$. The first key observation is that $B$ is uniquely determined by $b$ because

$$i_0 = \min(\{i \in [2n] \colon \; f(i) = 0\})\,.$$

The second key observation, which intervenes already in the proof of the first observation, is that the two words

$$b' = b_2 b_3 \dots b_{i_0 - 1} \text{ and } b'' = b_{i_0+1} b_{i_0+2} \dots b_{2n}$$

(if $i_0 = 2$ then $b' = \emptyset$, and if $i_0 = 2n$ then $b'' = \emptyset$) belong, after re-indexing them by $1, 2, \dots,$ to $A$, and that for every block $C \in P_b \setminus \{B\}$ we have $C \subset [2, i_0 - 1]$

or $C \subset [i_0+1, 2n]$. By induction, $P_{b'}$ and $P_{b''}$ are uniquely determined by $b'$ and $b''$, respectively. Thus also

$$P_b = \{B\} \cup (P_{b'} + 1) \cup (P_{b''} + i_0)\,,$$

where the meaning of the shifts should be clear, is uniquely determined by $b$. $\square$

Clearly, if $b, c \in A$, then both the concatenation $bc$ and the insertion $(b)$ belong to $A$.

For $b = b_1 b_2 \dots b_{2n} \in A$ with $n \in \mathbb{N}$ we set $s(b) = n$, and define $s(\emptyset) = 0$. So the size function $s(b)$ is the number of pairs of brackets. For example, $A_0 = \{\emptyset\}$ and

$$A_3 = \{()()(),\ ()(()),\ (())(),\ (()()),\ ((()))\}\,.$$

We denote $C_n = |A_n|$ and $C(x) = A(x)$, and derive an equation for the gf

$$C(x) = \textstyle\sum_{n \ge 0} C_n x^n\,.$$

We consider the map

$$f\colon A \setminus A_0 \to A \times A,\ \ f(b) = \langle b',\, b''\rangle\,,$$

defined by the familiar decomposition $b = (b')b''$. Here $(\dots)\dots$ is the first block (pair of brackets) in $b$. Note that $s(b) = 1 + s(b') + s(b'')$. It is easy to see that for every $n \in \mathbb{N}_0$,

$$f \mid A_{n+1}\colon A_{n+1} \to \textstyle\bigcup_{j=0}^{n} A_j \times A_{n-j}$$

is a bijection. Thus $C_0 = 1$ and for $n \in \mathbb{N}_0$,

$$C_{n+1} = \textstyle\sum_{j=0}^{n} C_j C_{n-j}\,.$$

This yields the equation

$$C(x) = 1 + xC(x)^2\,.$$

For a fps $S(x) = \sum_{n \ge 0} s_n x^n$ and $m \in \mathbb{N}_0$ we recall the notation $[x^m]S(x) = s_m$.

**Proposition 1.2** *For every $n \in \mathbb{N}_0$,*

$$|A_n| = C_n = [x^n]C(x) = [x^n]A(x) = -\frac{1}{2}\binom{1/2}{n+1}(-4)^{n+1} = \frac{1}{n+1}\binom{2n}{n}\,.$$

*Proof.* Solving the equation

$$xC(x)^2 - C(x) + 1 = 0$$

for $C(x)$ by the quadratic formula, we get

$$C(x) = \tfrac{1}{2x}\left(1 \pm \sqrt{1-4x}\right).$$

It is well known that

$$\sqrt{1-4x} = \textstyle\sum_{n \ge 0} \binom{1/2}{n}(-4x)^n \quad (\in \mathbb{C}[[x]])\,.$$

However, the rigorous formal justification (interpretation) of this equality is not, I think, well known; I postpone it to the section after the next section. Since $C(x) = 1 + x + x^2 + 5x^3 + \ldots$ contains no powers of $x$ with negative exponents, the correct choice of the sign is $-$ and $(m = n - 1)$

$$C(x) = \tfrac{1}{2x}\big(1 - \sqrt{1-4x}\big) = -\tfrac{1}{2}\textstyle\sum_{m\ge 0}\binom{1/2}{m+1}(-4)^{m+1}x^m\,.$$

Hence for every $n \in \mathbb{N}_0$,

$$\begin{aligned} C_n &= [x^n]C(x) = -\tfrac{1}{2}\tbinom{1/2}{n+1}(-4)^{n+1} \\ &= (-1)^{n+2}\tfrac{1}{2}\cdot\tfrac{1}{(n+1)!}\textstyle\prod_{i=0}^{n}\big(\tfrac{1}{2}-i\big)\cdot 4^{n+1} \\ &= \tfrac{1}{4}\cdot\tfrac{1}{(n+1)!}\textstyle\prod_{i=1}^{n}\big(i-\tfrac{1}{2}\big)\cdot 4^{n+1} \\ &= \tfrac{1\cdot 3\cdot 5\cdot\ldots\cdot(2n-1)}{(n+1)!\cdot 2^n}\cdot 4^n = \tfrac{(2n)!}{(n+1)!\cdot n!} = \tfrac{1}{n+1}\tbinom{2n}{n}\,. \end{aligned}$$

□

## Catalan numbers

The numbers

$$(C_n)_{n\ge 0} = \big(\tfrac{1}{n+1}\tbinom{2n}{n}\big)_{n\ge 0} = (1,\ 1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ 1430,\ 4862,\ \ldots)$$

are so called *Catalan numbers* [4, 6]. We know that $C_n$ is the number of proper bracketings with $n$ pairs of brackets. The book [6] lists literally hundreds of combinatorial interpretations of $C_n$.

Here is my favorite combinatorial, or rather geometrically-probabilistic, interpretation of $C_n$. We say that $n$ points $B_i = \langle x_i, y_i\rangle$ in the plane $\mathbb{R}^2$ have *convex position* if they form a convex $n$-gon. We say that they form a *convex chain* if, after they have been ordered so that

$$x_1 < x_2 < \cdots < x_n\,,$$

the $n-1$ vectors

$$B_i - B_{i-1},\ i = 2, 3, \ldots, n\,,$$

rotate in the counter-clockwise direction. If the points $B_i$ form a convex chain, then they have convex position, but there are many convex $n$-gons for $n \ge 3$ which do not form convex chains. The following conditional probability was computed 30 years ago in [7].

**Theorem 1.3 (Valtr, 1997)** *Let $n \in \mathbb{N}$ with $n \ge 2$ and let the $B_i$ be $n$ independent random points in the unit square* $[0,1]\times[0,1]$. *Then*

$$\mathrm{Prob}\big(\textit{the } B_i \textit{ form a convex chain} \mid \textit{the } B_i \textit{ have convex position}\big) = \frac{1}{C_{n-1}}\,.$$

In [7], this formula arises by dividing two separately calculated probabilities.

**Problem 1.4** *Prove Valtr's formula combinatorially by partitioning the event "the $B_i$ have convex position" into $C_{n-1}$ equiprobable events, one of them being the event "the $B_i$ form a convex chain".*

The case $n = 3$ is trivial, but already $n = 4$ seems challenging.

### Asymptotics of Catalan numbers

Let $n \in \mathbb{N}_0$. Then $0! = 1$ and $n! = \prod_{i=1}^{n} i$ for $n \geq 1$. Recall the *Stirling formula*

$$n! = (1 + o(1))\sqrt{2\pi n}\left(\tfrac{n}{\mathrm{e}}\right)^n \quad (n \to \infty)\,.$$

It has the following remarkably explicit form.

**Theorem 1.5 (Robbins [5])** *For every $n \in \mathbb{N}$,*

$$\sqrt{2\pi n} \cdot \left(\tfrac{n}{\mathrm{e}}\right)^n \cdot \mathrm{e}^{1/(1+12n)} \leq n! \leq \sqrt{2\pi n} \cdot \left(\tfrac{n}{\mathrm{e}}\right)^n \cdot \mathrm{e}^{1/12n}\,.$$

Thus, for example,

$$5039.33\ldots \leq 7! = 5040 \leq 5040.04\ldots\,.$$

**Corollary 1.6** *For every $n \in \mathbb{N}$,*

$$C_n = \frac{4^n}{\pi^{1/2}(n+1)n^{1/2}} \cdot \exp\delta\,,$$

*where* $-\frac{3}{24n+1} - \frac{1}{144n^2} \leq \delta \leq -\frac{3}{24n+2} + \frac{1}{144n^2}$.

*Proof.* By Theorem 1.5,

$$\begin{aligned} C_n &\geq (n+1)^{-1}\frac{(4\pi n)^{1/2}(2n/\mathrm{e})^{2n}}{\left((2\pi n)^{1/2}(n/\mathrm{e})^n\right)^2} \cdot \exp\left(\tfrac{1}{24n+1} - \tfrac{1}{6n}\right) \\ &= \tfrac{4^n}{\pi^{1/2}(n+1)n^{1/2}} \cdot \exp\left(\tfrac{-18n-1}{6n(24n+1)}\right) \\ &\geq \tfrac{4^n}{\pi^{1/2}(n+1)n^{1/2}} \cdot \exp\left(-\tfrac{3}{24n+1} - \tfrac{1}{144n^2}\right) \end{aligned}$$

Similarly,

$$\begin{aligned} C_n &\leq \tfrac{4^n}{\pi^{1/2}(n+1)n^{1/2}} \cdot \exp\left(\tfrac{1}{24n} - \tfrac{2}{12n+1}\right) \\ &= \tfrac{4^n}{\pi^{1/2}(n+1)n^{1/2}} \cdot \exp\left(\tfrac{-36n+1}{(12n+1)24n}\right) \\ &\leq \tfrac{4^n}{\pi^{1/2}(n+1)n^{1/2}} \cdot \exp\left(-\tfrac{3}{24n+2} + \tfrac{1}{144n^2}\right). \end{aligned}$$

□

Less precisely and less explicitly,

$$C_n \sim \pi^{-1/2}n^{-3/2}4^n \quad (n \to \infty)\,.$$

## 1.2 Basics on $R[[x]]$

Let $R = \langle R, 0_R, 1_R, +, \cdot\rangle$ be an (integral) domain; $R$ is a commutative ring with $1_R$ such that if $a, b \in R^*$ $(= R \setminus \{0_R\})$, then $ab = a \cdot b \in R^*$ as well. We define the set

$$R[[x]] = \{A\colon\ A \text{ is a map from } \mathbb{N}_0 \text{ to } R\}$$

of *formal power series*, briefly fps, over the domain $R$. The maps in $R[[x]]$ are standardly written as infinite formal linear combinations

$$A = A(x) = \textstyle\sum_{n\ge 0} a_n x^n$$

of powers of the formal variable $x$ with coefficients $a_n \in R$, so that $A(n) = A(x)(n) = a_n$. We use the notation $[x^n]A = [x^n]A(x) := a_n$.

Two special fps in $R[[x]]$ are

$$0 := 0_R x^0 + 0_R x^1 + 0_R x^2 + \ldots \ \text{ and } \ 1 := 1_R x^0 + 0_R x^1 + 0_R x^2 + \ldots .$$

We define the addition $+$ and multiplication $\cdot$ of two fps $A(x) = \sum_{n\ge 0} a_n x^n$ and $B(x) = \sum_{n\ge 0} b_n x^n$ by

$$A(x) + B(x) := \textstyle\sum_{n\ge 0}(a_n + b_n)x^n$$

and by

$$A(x)B(x) = A(x)\cdot B(x) := \textstyle\sum_{n\ge 0}\big(\sum_{j=0}^{n} a_j \cdot b_{n-j}\big)x^n$$

— this is called the *Cauchy product*. The *Hadamard product* is defined by

$$A(x) \odot B(x) := \textstyle\sum_{n\ge 0}(a_n \cdot b_n)x^n .$$

**Proposition 1.7** $R[[x]] = \langle R[[x]], 0, 1, +, \cdot\rangle$ *is an integral domain.*

*Proof.* We leave the verification of the ring axioms as an exercise for the reader and only prove the integrality. Let $A(x) = a_m x^m + a_{m+1}x^{m+1} + \ldots$ and $B(x) = b_n x^n + b_{n+1}x^{n+1} + \ldots$ be two nonzero fps, so that $m, n \in \mathbb{N}_0$ and $a_m, b_n \neq 0_R$. Then

$$A(x)B(x) = a_m b_n x^{m+n} + c_{m+n+1}x^{m+n+1} + c_{m+n+2}x^{m+n+2} + \ldots$$

with $c_j \in R$ and $a_m b_n \neq 0_R$ because $R$ is a domain. So $A(x)B(x) \neq 0$. □

**Proposition 1.8** *We have*

$$R[[x]]^\times = \{A(x) \in R[[x]]\colon\ [x^0]A(x) \in R^\times\} .$$

*Proof.*

□

For $A \in R[[x]]^*$ we define the *order* $\mathrm{ord}(A)$ of $A$ to be the minimum $n \in \mathbb{N}_0$ such that $[x^n]A \neq 0_R$. We set $\mathrm{ord}(0) := +\infty$.

**Corollary 1.9** *For every two fps $A, B \in R[[x]]$,*

$$\mathrm{ord}(AB) = \mathrm{ord}(A) + \mathrm{ord}(B) \quad (\in \mathbb{N}_0^*)\,.$$

*Proof.*

□

**Proposition 1.10** $\langle R[[x]], \|\cdot\|\rangle$ *is a complete non-Archimedean normed domain.*

*Proof.*

□

## 1.3 Why $\sqrt{1-4x} = \sum_{n\geq 0}\binom{1/2}{n}(-4)^n x^n$?

**Theorem 1.11** *Let $A(x) = ux^{2m}(1+t_1x+t_2x^2+\dots)$ with $m \in \mathbb{N}_0$ and $u, t_j \in \mathbb{C}$ be a fps in $\mathbb{C}[[x]]$ and $v \in \mathbb{C}$ be such that $v^2 = u$. Then*

$$\left(vx^m \sum_{n\geq 0}\binom{1/2}{n}(t_1x + t_2x^2 + \dots)^n\right)^2 = A(x)\,.$$

*Proof.*

□

## 1.4 Pólya's theorem via the symbolic method

# Chapter 2

# Numbers of SAWs in the hexagonal grid

# Chapter 3

# Combinatorics and Complexity of Partition Functions

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