Combinatorial Counting 2025: the Ramsey theorem for pairs and the Catalan numbers

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(lecture notes for the course taught in summer term 2025)

Contents

Introduction

These lecture notes

Notation. We use $\equiv$ as definitional equality; in $x \equiv y$ the new symbol x is being defined by the already known expression y. Sometimes x and y exchange their roles. Recall that $f : X \to Y$, where X and Y are sets, means that f is a map (function) from X to Y. So f is a set such that $f \subset X \times Y$ and for every $x \in X$ there is a unique $y \in Y$ for which $(x, y) \in f$, which is standardly written as $f(x) = y$. Let $f : X \to Y$ and Z be any set. The *restriction* of f to Z is the map $f \,|\, Z : X \cap Z \to Y$ with values $(f \,|\, Z)(x) \equiv f(x)$, $x \in X \cap Z$. Often we write instead of $f \,|\, Z$ just f. The *image* and the *preimage* of Z by f is the respective set

$$f[Z] \equiv \{f(x) :\ x \in X \cap Z\} \ \ (\subset Y) \ \text{ and } \ f^{-1}[Z] \equiv \{x \in X :\ f(x) \in Z\} \ \ (\subset X)\,.$$

If $Z = \{z\}$ is a singleton, we usually write (in analogy with $f(x)$) instead of $f^{-1}[\{z\}]$ just $f^{-1}[z]$.

We denote by $\mathbb{N} = \{1, 2, \dots\}$ the (infinite) set of natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ are nonnegative integers. For $n \in \mathbb{N}$ we define $[n] \equiv \{1, 2, \dots, n\}$; we set $[0] \equiv \emptyset$. For any finite set X we denote by $|X|$ ($\in \mathbb{N}_0$) the number of its elements. For any set X and $k \in \mathbb{N}_0$,

$$\tbinom{X}{k} \equiv \{Y :\ Y \subset X,\, Y \text{ is finite and } |Y| = k\}\,.$$

Thus $\binom{X}{0} = \{\emptyset\}$ and $\binom{X}{1} = \{\{x\} :\ x \in X\}$ ($\neq X$). We have $\binom{X}{k} = \emptyset$ whenever X is finite and $|X| < k$. Also, for finite X the set $\binom{X}{k}$ is finite too and for $0 \le k \le |X|$ we have equalities

$$\left|\tbinom{X}{k}\right| = \tbinom{|X|}{k} = \tfrac{|X|(|X|-1)\dots(|X|-k+1)}{k!}\,.$$

Mostly we work with the sets $\binom{X}{2}$ and call their elements *edges*.

Chapter 1

The Ramsey theorem for pairs: simple bounds

Theorem 1.1 (Ramsey, 1930) *Let $r, p, k \in \mathbb{N}$. Then there is an $n \in \mathbb{N}$ such that for every map $\chi : \binom{[n]}{p} \to [r]$ there is a k-element set $Y \subset [n]$ for which the restriction $\chi \mid \binom{Y}{p}$ is constant.*

In Chapters 1 and 2 we deal with the function $R_r(k) : \mathbb{N}^2 \to \mathbb{N}$ corresponding to the pairs case $p = 2$ of the theorem. Its values are called *Ramsey numbers (for pairs)* and it is defined as follows.

Definition 1.2 *Let $r, k \in \mathbb{N}$. We define $R_r(k)$ to be the minimum $n \in \mathbb{N}$ such that for every union $\binom{[n]}{2} = \bigcup_{i=1}^{r} X_i$ there exists an index $i \in [r]$ and a k-element set $Y \subset [n]$ such that $\binom{Y}{2} \subset X_i$.*

In Proposition 1.5 we prove that $R_r(k)$ is defined for every r and k. Without loss of generality the sets X_i may be assumed to be pairwise disjoint and to form a partition (of $\binom{[n]}{2}$). This follows from the next proposition.

Proposition 1.3 *Let $r \in \mathbb{N}$, X be a set and $\{Y_i : i \in [r]\}$ be a set system such that $X = \bigcup_{i=1}^{r} Y_i$. Then there is a set system $\{Z_i : i \in [r]\}$ with the following properties.*

1. *$Z_i \subset Y_i$ for every $i \in [r]$.*
2. *$X = \bigcup_{i=1}^{r} Z_i$.*
3. *$Z_i \cap Z_j = \emptyset$ for every $i, j \in [r]$ with $i \neq j$.*

Proof. Let $Y_0 \equiv \emptyset$ and $Z_i \equiv Y_i \setminus \bigcup_{j=0}^{i-1} Y_j$, $i \in [r]$. It is not hard to see that this set system $\{Z_i : i \in [r]\}$ has the three stated properties. □

An exercise for the reader is to extend this proposition to infinite set systems.

So in other words, $R_r(k)$ is the minimum $n \in \mathbb{N}$ such that for every *coloring* (map) $\chi : \binom{[n]}{2} \to [r]$ there is a set $Y \subset [n]$ with $|Y| = k$ for which the restriction $\chi \,|\, \binom{Y}{2}$ is a constant map. We say that the set Y is *χ-homogeneous.* Yet another equivalent definition of $R_r(k)$ is that it is the minimum $n \in \mathbb{N}$ such that for every r-coloring of edges in the complete graph $K_n = (n, \binom{[n]}{2})$ there is a monochromatic k-clique.

The following *pigeonhole principles*, a finite and an infinite one, are Ramsey theorems for *singletons* (1-element sets). They were known, of course, long time before Theorem 1.1.

Proposition 1.4 (pigeonhole principles) *Let $k, r \in \mathbb{N}$. The following holds.*

1. *If $n \equiv r(k-1)+1$, X is any finite set with $|X| = n$ and $\chi : X \to [r]$ is any map, then there is an $i \in [r]$ such that $|\chi^{-1}(i)| \ge k$.*
2. *If X is any infinite set and $\chi : X \to [r]$ is any map, then there is an $i \in [r]$ such that the set $\chi^{-1}(i)$ is infinite.*

Proof. 1. Since $X = \bigcup_{i \in [r]} \chi^{-1}(i)$ is a partition, if there were no such i then we would have the contradiction

$$n = |X| = \textstyle\sum_{i \in [r]} |\chi^{-1}(i)| \le \sum_{i \in [r]} (k-1) = r(k-1)\,.$$

2. If there were no such i, we would have the contradiction that the set $X = \bigcup_{i=1}^{r} \chi^{-1}(i)$ is finite as it is a finite (disjoint) union of finite sets. □

1.1 Finite case

In this section we obtain several elementary upper and lower bounds on the Ramsey numbers $R_r(k)$ which were introduced in Definition 1.2. First we show that $R_r(k)$ is defined for every $r, k \in \mathbb{N}$. If $\chi : \binom{[n]}{2} \to [r]$ and $Y \subset [n]$, we say that the set Y is *χ-min-homogeneous* if for every $e, f \in \binom{Y}{2}$ it holds that $\chi(e) = \chi(f)$ iff $\min e = \min f$. In the next Section 1.2 we consider a generalization of this kind of colorings.

Proposition 1.5 *Let $r, k \in \mathbb{N}$. Then $R_1(k) = k$, $R_r(1) = 1$ and for $r, k \ge 2$ we have the bound $R_r(k) \le r^{rk-2}$.*

Proof. The cases when $r = 1$ or $k = 1$ are clear. Let $r, k \ge 2$. We show, for the coloring form of Definition 1.2, that $n \equiv r^{rk-2}$ works. Let $\chi : \binom{[n]}{2} \to [r]$ be any map. We set $l \equiv r(k-1)+1$ and define sets $A_0 \equiv [n]$, A_1, $\dots$, A_{l-1} such that $A_0 \supset A_1 \supset \dots \supset A_{l-1} \ne \emptyset$, $\min A_0 = 1 < \min A_1 < \dots < \min A_{l-1}$, that for every $i \in [l-1]$ the edges $\{\min A_{i-1}, x\}$, $x \in A_i$, have in χ the same color, and that $|A_i| = r^{rk-2-i}$, $i = 0, 1, \dots, l-1$. Suppose that $i \in [l-1]$ and that the sets A_0, A_1, $\dots$, A_{i-1} with the stated properties are already defined. By 1 of Proposition 1.4, at least $\lceil \frac{|A_{i-1}|-1}{r} \rceil = \lceil \frac{r^{rk-2-i+1}-1}{r} \rceil = r^{rk-2-i}$ edges $\{\min A_{i-1}, x\}$ with $x \in A_{i-1} \setminus \{\min A_{i-1}\}$ have in χ the same color; we define A_i to be some r^{rk-2-i} endpoints x of such edges. Thus we have sets A_0, A_1, $\ldots$, A_{l-1} with the stated properties; note that $A_{l-1} \neq \emptyset$ because $rk - 2 - (l-1) = r - 2 \geq 0$. We consider the l-element set

$$X \equiv \{\min A_{i-1} :\ i \in [l]\} .$$

It follows that X is χ-min-homogeneous and we can define the map $\psi : X \to [r]$ by setting $\psi(x) \equiv \chi(e)$ for any $e \in \binom{X}{2}$ with $\min e = x$; for $x = \max X$ when there is no such e we define $\psi(x)$ arbitrarily. By 1 of Proposition 1.4 there is a set $Y \subset X$ such that $|Y| = k$ and $\psi \,|\, Y$ is constant. It follows that Y is the sought for k-element χ-homogeneous set. $\Box$

In the next chapter we use this bound in the slightly weaker but simpler form $R_r(k) \leq r^{rk}$. Thus in the simplest nontrivial case $r = p = 2$ of Theorem 1.1 we have the following upper bound.

Corollary 1.6 *For every* $k \in \mathbb{N}$,

$$R_2(k) \leq 4^{k-1} .$$

1.2 The canonical Ramsey theorem

If $k \in \mathbb{N}$, $e = \{e_1, e_2, \ldots, e_k\}_<$ in $\binom{\mathbb{N}}{k}$ is a k-element set of natural numbers with the elements e_i listed increasingly and if $I \subset [k]$, we define

$$e : I \equiv \{e_i :\ i \in I\} .$$

If $X \subset \binom{\mathbb{N}}{k}$ and $\chi \colon X \to \mathbb{N}$ is any coloring of X (by infinitely many colors), then we call χ *canonical*, or more precisely *I-canonical*, if there is a set $I \subset [k]$ such that for every $e, f \in X$,

$$\chi(e) = \chi(f) \iff e : I = f : I .$$

This section is devoted to the function $\mathrm{ER}(k; l) \colon \mathbb{N}^2 \to \mathbb{N}$, especially for $k = 2$, defined as follows.

Definition 1.7 *Let* $k, l \in \mathbb{N}$. *Then* $\mathrm{ER}(k; l)$ *is the minimum* $n \in \mathbb{N}$ *such that for every coloring* $\chi : \binom{[n]}{k} \to \mathbb{N}$ *there exists an* l*-element set* $Y \subset [n]$ *such that the restriction* $\chi \,|\, \binom{Y}{k}$ *is canonical. We set* $\mathrm{ER}(l) \equiv \mathrm{ER}(2; l)$.

In 1950 P. Erdős and R. Rado proved in [3] that the numbers $\mathrm{ER}(k; l)$ exist for every $k, l \in \mathbb{N}$. For $k = 1$ these numbers are easily determined exactly.

Proposition 1.8 $\mathrm{ER}(1; l) = (l-1)^2 + 1$ *for every* $l \in \mathbb{N}$.

Proof. Let $l \in \mathbb{N}$, $n \equiv (l-1)^2 + 1$ and $\chi\colon [n] \to \mathbb{N}$. Since $n = \sum_{i\in\mathbb{N}} |\chi^{-1}(i)|$, we see that there is a set $X \subset [n]$ such that $|X| = l$ and $\chi \,|\, X$ is constant or 1-1 (injective). Thus

$$\mathrm{ER}(1;\, l) \le n = (l-1)^2 + 1\,.$$

On the other hand, if we set $n \equiv (l-1)^2$ and, for $i = 1, 2, \dots, l-1$ and $(i-1)(l-1) < j \le i(l-1)$, define the coloring $\chi\colon [n] \to \mathbb{N}$ by $\chi(j) \equiv i$, we get the bound $\mathrm{ER}(1; l) > n = (l-1)^2$; for this χ there is no l-element canonical set (for $k = 1$). Thus we get the stated equality. □

In 1996 S. Shelah proved in [10] for any $k \ge 2$ a strong general upper bound on $\mathrm{ER}(k; l)$ in the form of an iterated $(k-1)$-fold exponential. For $k, l \in \mathbb{N}$ we set $\mathrm{tow}(1; l) \equiv 2^l$ and $\mathrm{tow}(k; l) \equiv 2^{\mathrm{tow}(k-1;l)}$ for $k \ge 2$.

Theorem 1.9 (Shelah, 1996) *There is a constant $c > 0$ such that for every $k, l \in \mathbb{N}$ with $k \ge 2$,*

$$\mathrm{ER}(k;\, l) \le \mathrm{tow}\big(k-1;\, cl^{8(2k-1)}\big)\,.$$

In the rest of this section we prove two theorems on $\mathrm{ER}(l) = \mathrm{ER}(2; l)$. We begin with a theorem due to H. Lefmann and V. Rödl. They obtained the easy lower bound in [6], and the harder to prove upper bound in [7].

Theorem 1.10 (Lefmann and Rödl, 1993 and 1995) *For some constants $c_1, c_2 > 0$ and every $l \in \mathbb{N}$ with $l \ge 2$,*

$$2^{c_1 l^2} \le \mathrm{ER}(l) \le 2^{c_2 l^2 \log l}\,.$$

We begin with the lower bound.

The lower bound $2^{c_1^2 l} \le \mathrm{ER}(l)$

We prove the lower bound of Lefmann and Rödl and begin with a lemma.

Lemma 1.11 *Let $k, l \in \mathbb{N}$ with $k \le l$ and $I \subset [k]$ with $I \ne \emptyset$. Then there exist $l-k+1$ sets $X_i \in \binom{[l]}{k}$, $i \in [l-k+1]$, such that the $l-k+1$ sets*

$$X_i : I,\ \ i \in [l-k+1]\,,$$

are mutually distinct

Proof. For $i = 0, 1, \dots, l-k$ set $X_i \equiv \{i+1, i+2, \dots, i+k\}$. □

Recall that for $t, k, l \in \mathbb{N}$ the (classical) Ramsey number $\mathrm{R}_t(k; l)$ is the minimum $n \in \mathbb{N}$ such that for every $\chi\colon \binom{[n]}{k} \to [t]$ there is an l-element set $X \subset [n]$ such that the restriction $\chi \,|\, \binom{X}{k}$ is constant.

Proposition 1.12 *For every $k, l \in \mathbb{N}$ with $k < l$,*

$$\mathrm{ER}(k;\, l) \ge \mathrm{R}_{l-k}(k;\, l)\,.$$

Proof. Let k and l be as stated, $n \equiv \mathrm{R}_{l-k}(k;l) - 1$ and let $\chi\colon \binom{[n]}{k} \to [l-k]$ be such that there is no l-element χ-monochromatic set. But then there is also no $I \subset [k]$ and no l-element set $X \subset [n]$ such that $\chi \,|\, \binom{X}{k}$ is I-canonical. For $I = \emptyset$ it follows from the non-existence of monochromatic set, and for $I \neq \emptyset$ it follows from Lemma 1.11 which shows that then at least $l - k + 1$ distinct colors would be needed. Hence $\mathrm{ER}(k;l) > n$ and we get the stated inequality. □

This concludes the proof of the lower bound.

The upper bound $2^{c_2 l^2 \log l} \geq \mathrm{ER}(l)$

We prove the upper bound of Lefmann and Rödl.

1.3 Infinite case

Theorem 1.13 (infinite Ramsey for pairs) *Let $r \in \mathbb{N}$. Then for every map $\chi\colon \binom{\mathbb{N}}{2} \to [r]$ there is an infinite χ-homogeneous set, an infinite set $Y \subset \mathbb{N}$ such that $\chi \,|\, \binom{Y}{2}$ is constant.*

Proof. Let r and χ be as stated. We define a sequence of infinite sets $A_0 \equiv \mathbb{N}$, $A_1, \ldots$ such that $A_0 \supset A_1 \supset \ldots$, $\min A_0 = 1 < \min A_1 < \ldots$ and that for every n the pairs $\{\min A_{n-1}, x\}$, $x \in A_n$, have in χ the same color. Suppose that $n \in \mathbb{N}$ and that the sets $A_0, A_1, \ldots, A_{n-1}$ with the stated properties are already defined. By 2 of Proposition 1.4, for infinitely many $x \in A_{n-1} \setminus \{\min A_{n-1}\}$ the edges $\{\min A_{n-1}, x\}$ have in χ the same color; we define A_n as the set of these numbers x. Thus we get a sequence of sets $(A_n)_{n\geq 0}$ with the stated properties. We define the infinite set

$$X \equiv \{\min A_{n-1} :\ n \in \mathbb{N}\}\,.$$

It follows that X is χ-min-homogeneous. As before we can define the map $\psi : X \to [r]$ by setting $\psi(x) \equiv \chi(e)$ for any $e \in \binom{X}{2}$ with $\min e = x$. By 2 of Proposition 1.4 there is an infinite set $Y \subset X$ such that $\psi \,|\, Y$ is constant. It follows that Y is an infinite χ-homogeneous set. □

Theorem 1.14 (compactness) *Let $r \in \mathbb{N}$. For every sequence (χ_n) of colorings $\chi_n : \binom{[n]}{2} \to [r]$ there exists a coloring $\chi : \binom{\mathbb{N}}{2} \to [r]$ with the following property. For every k there is an n, $n \geq k$, with*

$$\chi_n \,|\, \tbinom{[k]}{2} = \chi \,|\, \tbinom{[k]}{2}\,.$$

Proof. Let r and χ_n, $n \in \mathbb{N}$, be as stated. Let $F : \mathbb{N} \to \binom{\mathbb{N}}{2}$ be a bijection, thus the sequence $F(1)$, $F(2)$, $\ldots$ enumerates the edges of the countable complete graph $K_{\mathbb{N}}$. We define by induction on $j = 0, 1, \ldots$ infinite sets A_j such that $A_0 = \mathbb{N}$, $A_0 \supset A_1 \supset \ldots$ and that if $A_j = \{a_{1,j} < a_{2,j} < \ldots\}$, then for every $i \in [j]$ the values $\chi_{a_{1,j}}(F(i))$, $\chi_{a_{2,j}}(F(i))$, $\dots$ are all defined (that is, $a_{1,j} \ge \max F(i)$) and are all equal. In other words, for every $i \in [j]$ we have $|\{\chi_a(F(i)) : \ a \in A_j\}| = 1$. Suppose that $j \in \mathbb{N}$ and that the sets A_0, A_1, $\dots$, A_{j-1} (with the stated properties) are already defined. We define A_j as any infinite subset of A_{j-1} for which every value $\chi_a(F(j))$, $a \in A_j$, is defined and $|\{\chi_a(F(j)) : \ a \in A_j\}| = 1$. Such a subset exists by 2 of Proposition 1.4. Thus we get the sequence of sets $(A_j)_{j\ge 0}$ with the stated properties. We define the map $\chi : \binom{\mathbb{N}}{2} \to [r]$ for $e \in \binom{\mathbb{N}}{2}$ by setting, with $j \equiv F^{-1}(e)$,

$$\chi(e) \equiv \chi_a(e) \text{ for any } a \in A_j \,.$$

It is clear that this definition is correct — the color $\chi_a(e)$ does not depend on the element $a \in A_j$ — and we show that χ has the stated property. So let $k \in \mathbb{N}$. We take a $j \in \mathbb{N}$ such that $F[\,[j]\,] \supset \binom{[k]}{2}$ and take any $n \in A_j$. Let $e \in \binom{[k]}{2}$. Then by the definition of A_j and χ we have with $i \equiv F^{-1}(e)$ that $n \ge k$, $i \in [j]$, $n \in A_i$ and thus $\chi(e) = \chi_n(e)$, as required. □

We say that a finite set $X \subset \mathbb{N}$ is *big* if $|X| \ge \min X$.

Theorem 1.15 (big Ramsey for pairs) *Let $r \in \mathbb{N}$. Then for every k there is an n such that for every coloring $\chi : \binom{[n]}{2} \to [r]$ there exists a big and at least k-element χ-homogeneous set $Y \subset [n]$.*

Proof. Let $r, k \in \mathbb{N}$. Suppose for the contrary that for every n there is a coloring $\chi_n : \binom{[n]}{2} \to [r]$ that has no big and at least k-element χ_n-homogeneous set. It follows that the same holds for the coloring $\chi : \binom{\mathbb{N}}{2} \to [r]$ obtained from the sequence (χ_n) in Theorem 1.14. But this is a contradiction because it is easy to deduce from Theorem 1.13 that every r-coloring of $\binom{\mathbb{N}}{2}$ has a big and at least k-element homogeneous set. Indeed, if $\psi : \binom{\mathbb{N}}{2} \to [r]$ is any coloring and $\{a_1 < a_2 < \dots\} \subset \mathbb{N}$ is the infinite ψ-homogeneous set provided by Theorem 1.13, then

$$\{a_1 < a_2 < \cdots < a_{k+a_1}\}$$

is a big and at least k-element ψ-homogeneous set. □

Theorem 1.16 (Erdős–Dushnik–Miller) *Suppose that κ is an infinite cardinal. Then for every partition $\binom{\kappa}{2} = A \cup B$ there exists a set $C \subset \kappa$ such that $|C| = \omega$ and $\binom{C}{2} \subset A$, or $|C| = \kappa$ and $\binom{C}{2} \subset B$.*

Proof.

□

Chapter 2

The Ramsey theorem for pairs: the BBCGHMST bound

Recall that for $r, k \in \mathbb{N}$ we denote by $R_r(k)$ the minimum $n \in \mathbb{N}$ such that for every $\chi\colon \binom{[n]}{2} \to [r]$ there is a k-element monochromatic set: a set $X \subset [n]$ such that $|X| = k$ and the restriction $\chi \,|\, \binom{X}{2}$ is constant. More generally, for $r, k_1, \ldots, k_r \in \mathbb{N}$ we denote by $R_r(k_1, \ldots, k_r)$ the minimum $n \in \mathbb{N}$ such that for every $\chi\colon \binom{[n]}{2} \to [r]$ there exists an $i \in [r]$ and a k_i-element i-monochromatic set: a set $X \subset [n]$ such that $|X| = k_i$ and the restriction $\chi \,|\, \binom{X}{2}$ is constantly i.

If $r \in \mathbb{N}$, $V \subset \mathbb{N}$ is finite, $\chi\colon \binom{V}{2} \to [r]$, $u \in V$ and $i \in [r]$, we denote by

$$N_i(u) \equiv \{v \in V\colon\ \chi(\{u, v\}) = i\}$$

the "neighborhood of u in colour i". If $A, B \subset \mathbb{N}$ are finite and disjoint, we call the graph

$$H = \big(A \cup B,\ \tbinom{A}{2} \cup \{\{a, b\}\colon\ a \in A,\ b \in B\}\big)$$

the (A, B)-*book*. If $|A| = t$ and $|B| = m$, we say that H is the (t, m)-*book*. Books are key graphs in the proof of Theorem 2.1 below. The theorem and its proof are taken from the preprint

> Paul Balister, Béla Bollobás, Marcelo Campos, Simon Griffiths, Eoin Hurley, Robert Morris, Julian Sahasrabudhe and Marius Tiba: Upper bounds for multicolour Ramsey numbers, arXiv:2410.17197v1, 17 pp., October 2024.

Theorem 2.1 (T. 5.1) *Let $r \in \mathbb{N}$ with $r \geq 2$ and $\delta \equiv 2^{-160} r^{-12}$. Then for every $k \in \mathbb{N}$ with $k \geq 2^{160} r^{16}$ we have*

$$R_r(k) \leq \mathrm{e}^{-\delta k} r^{rk}\,.$$

2.1 The proof of Theorem 2.1, modulo the proof of Theorem ??

Proof of Theorem 2.1. We follow [1], but we rearrange the proof. Let $r \geq 2$ and $\delta \equiv 2^{-160}r^{-12}$. Let $k \geq 2^{160}r^{16}$ and $n \geq \mathrm{e}^{-\delta k}r^{rk}$. Let $\chi\colon \binom{[n]}{2} \to [r]$. Let $\varepsilon \equiv 2^{-50}r^{-4}$ and let $S_1, \ldots, S_r$ and W be the sets from Lemma 2.2.

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Lemma 2.2 (L. 5.2) *Let $n, r \in \mathbb{N}$, $\varepsilon > 0$ and $\chi\colon \binom{[n]}{2} \to [r]$. Then there exist mutually disjoint sets $S_1, \ldots, S_r, W \subset [n]$, where possibly $S_i = \emptyset$, such that the following holds.*

1. $|W| \geq n \cdot \left(\frac{1+\varepsilon}{r}\right)^{|S_1|+\cdots+|S_r|}$.
2. *For any $i \in [r]$ and $w \in W$ we have $|N_i(w) \cap W| \geq (r^{-1} - \varepsilon)(|W| - 1)$.*
3. *For any $i \in [r]$, the (S_i, W)-book is i-monochromatic in χ.*

We prove this lemma later in this section.

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(We continue the proof of Theorem 2.1.) In the first case $|S_1| + \cdots + |S_r| \geq \varepsilon^2 k$. Then by Lemma 2.3

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Lemma 2.3 (L. 5.3) *Let $k, r \in \mathbb{N}$ with $r \geq 2$ and $\varepsilon \in (0, 1)$. Then for every $s_1, \ldots, s_r \in [k]$ with $s \equiv s_1 + \cdots + s_r \geq \varepsilon^2 k$ we have*

$$R_r(k - s_1, \ldots, k - s_r) \leq \mathrm{e}^{-\varepsilon^3 k/2} \cdot \left(\frac{1+\varepsilon}{r}\right)^s \cdot r^{rk}.$$

Proof.

□

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(we continue the proof of Theorem 2.1) and our lower bounds on n and $|W|$ we have $|W| \geq$

2.2 The proof of Theorem ??

Chapter 3

The Catalan numbers

Bibliography

[1] P. Balister, B. Bollobás, M. Campos, S. Griffiths, E. Hurley, R. Morris, J. Sahasrabudhe and M. Tiba, Upper bounds for multicolour Ramsey numbers, arXiv:2410.17197v1, 2024, 17 pp.

[2] M. Campos, S. Griffiths, R. Morris and J. Sahasrabudhe, An exponential improvement for diagonal Ramsey, arXiv:2303.09521v1, 2023, 57 pp.

[3] P. Erdős and R. Rado, A combinatorial theorem, *J. London Math. Soc.* **25** (1950), 249–255

[4] T. Jech, *Set Theory. The Third Millennium Edition, revised and expanded*, Springer-Verlag, Berlin 2003

[5] M. Klazar, On a proof of Ramsey theorem and of Erdös-Rado theorem for pairs, preprint KAM Series, 93–256, 1993, 5 pp.

[6] H. Lefmann and V. Rödl, On canonical Ramsey numbers for complete graphs versus paths, *J. Combin. Theory* Ser. B **58** (1993), 1–13

[7] H. Lefmann and V. Rödl, On Erdős-Rado numbers, *Combinatorica* **15** (1995), 85–104

[8] Ch. Misak, *Frank Ramsey. A Sheer Excess of Powers*, Oxford University Press, Oxford 2020

[9] L. Paulson, Formalising new mathematics in Isabelle: diagonal Ramsey, arXiv:2501.10852v1, 2025, 22 pp.

[10] S. Shelah, Finite canonization, *Comment. Math. Univ. Carolin.* **37** (1996), 445–456.